Math 132, Lecture 1: Trigonometry

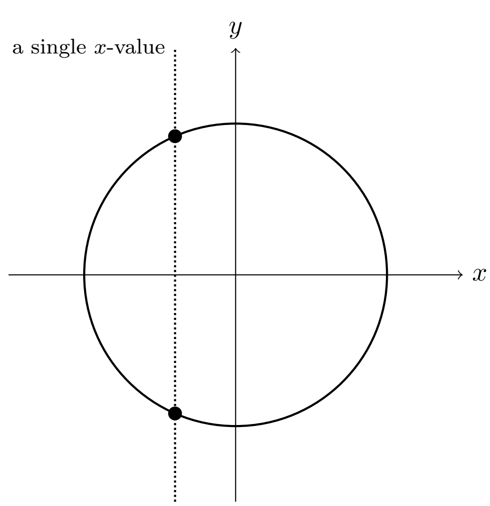
Charles Staats

Wednesday, 4 January 2012

1 The circular functions: Definitions of sine and cosine

Any time we trace a path in the plane, we implicitly define two functions of time. The first function takes t to the x-coordinate of our path at time t; the second function takes t to the y-coordinate. Specifying x and y as functions of t is called *parametric* graphing. It is more versatile than simply plotting y as a function of x.

The particular path we are interested in at the moment is the unit circle, i.e., the circle of radius 1 centered at the origin. The circle is *not* the graph of a function, since a single x-coordinate can have more than one corresponding y-coordinate:

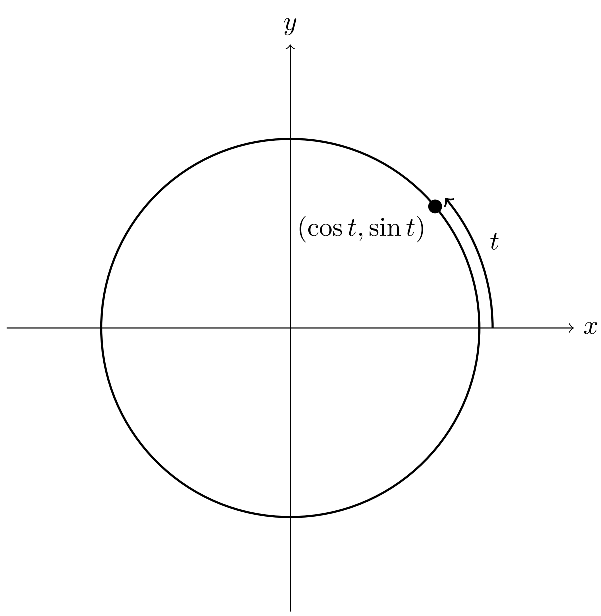


Nevertheless, the circle can be described, parametrically, by two functions. By convention, we start at the rightmost point $(1, 0)$ and then move counterclockwise around the circle at constant speed 1. Then for any given time t, there is a unique x-coordinate $x(t)$ and a unique y-coordinate $y(t)$, so we have two functions.

It is perhaps natural to ask whether these functions can be described by formulas. So far in this class, we have used only $+$, $-$, $\times$, $\div$, and exponents and roots. As it turns out, no finite-length expression using only these operations can describe either of the two functions $x(t)$ and $y(t)$. Nevertheless, we know that these two functions do exist. So, we invent new symbols, sin (sine) and cos (cosine), to describe them.

Definition. Let t be a real number. We define $\sin t$ and $\cos t$ as follows: Let $(x(t), y(t))$ be the point on the unit circle obtained by starting at the point $(1, 0)$ and traveling a distance of t in the counterclockwise direction. Then we define

$$\sin t = y(t)$$
$$\cos t = x(t).$$

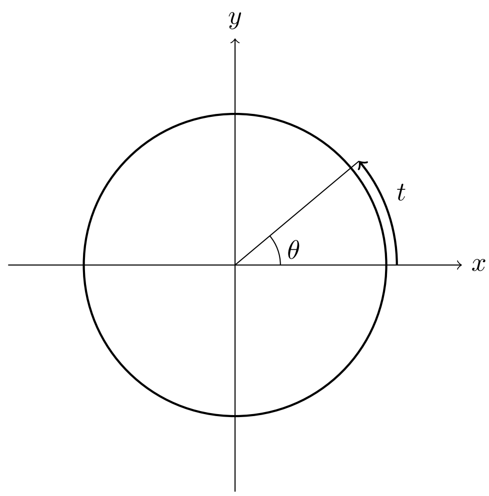


Note: To make this apply for negative t, we take, e.g., "traveling -2 units counterclockwise" to mean "traveling 2 units clockwise."

This is the same idea as defining the $\sqrt{\ }$ symbol to solve problems that otherwise would have no formula for the solution.

2 Radians

Whenever we specify a distance t to travel along the circle, there is a corresponding angle θ:



If we divide t by the circumference of the unit circle[1], we obtain what fraction of the circle t represents. If we measure θ in degrees, and divide it by the total number of degrees in the circle (360), then we obtain the same fraction of the circle. In other words,

$$\frac{t}{2\pi} = \frac{\theta}{360}.$$

Thus, t is really a measure of *angle*, not just distance; only, in making this measurement, we are dividing the circle into 2π parts rather than 360 parts.

It may seem odd to divide the circle into 360 parts, but it seems even odder to divide it into 2π parts—this is not even an integer. Why do we do this? Essentially, we use radians because calculus formulas look *much* nicer in radians than in degrees. For instance, let θ be the measure of an angle in degrees, and let t be the measure of the same angle in radians. Then, as we will see later, the following equations are all true.

radians	degrees
$\lim_{t\to 0} \frac{\sin t}{t} = 1$	$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = \frac{\pi}{180}$
$\frac{d}{dt}\sin t = \cos t$	$\frac{d}{d\theta}\sin\theta = \frac{\pi}{180}\cos\theta$

As you can see, the equations in radians are much nicer than the equations in degrees. When we prove the first statement, we will perhaps see why.

[1] The circumference of the unit circle is $2\pi r = 2\pi \cdot 1 = 2\pi$.

There's one more pair of very nice formulas, important if we actually want to compute sine and cosine:

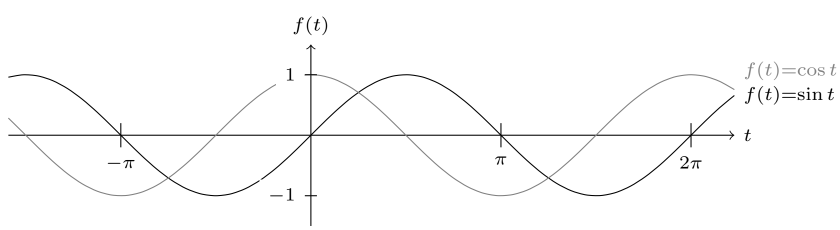
$$\sin t = t - \tfrac{1}{3!}t^3 + \tfrac{1}{5!}t^5 - \tfrac{1}{7!}t^7 + \cdots,$$
$$\cos t = 1 - \tfrac{1}{2!}t^2 + \tfrac{1}{4!}t^4 - \tfrac{1}{6!}t^6 + \cdots.$$

These formulas are called the *Taylor series* for sine and cosine; we will get some idea why they work next quarter. If we have measured t in radians and we want to get a decimal approximation for $\sin t$ or $\cos t$, we can usually get a very good approximation by computing just the first few terms of the Taylor series. If we have measured the angle in degrees, we probably want to convert it to radians so that we can apply these formulas.

Warning. *The Taylor series formulas above will (almost?) never come up again until next quarter. In particular, you should not use them to solve any problems unless specifically told otherwise.*

3 Graphs of sine and cosine

Although we probably won't have time to discuss these until next lecture, I would be remiss if I did not at least show/remind you what the graphs of sine and cosine look like.



Assignment 1 (due Friday, 6 January)

Note: I will sometimes assign homework problems that are supposed to prepare you for the next lecture, rather than review the previous one. In this case, you may have to consult the textbook to figure out how to do them.

Section 0.7, Problems 1(a)–(c), 2(a)–(c), 9, and 14.

Explain why the Taylor series formulas (Lecture 1, top of page 4) do not contradict the assertion at the top of page 2 that the sine and cosine cannot be given by non-trigonometric. Hint: You really do want to look at page 2 to see exactly what this assertion is.

Assignment 2 (due Monday, 9 January)

Section 0.7, Problems 1(d)–(f), 2(d)–(f), 16, and 17.

Section 1.4, Problems 1, 2, 15, and 16.

Math 132, Lecture 2: Trigonometry

Charles Staats

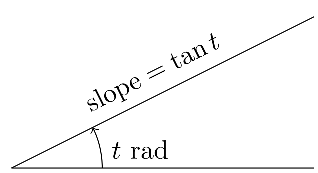
Friday, 6 January 2012

1 Slope versus angle of inclination

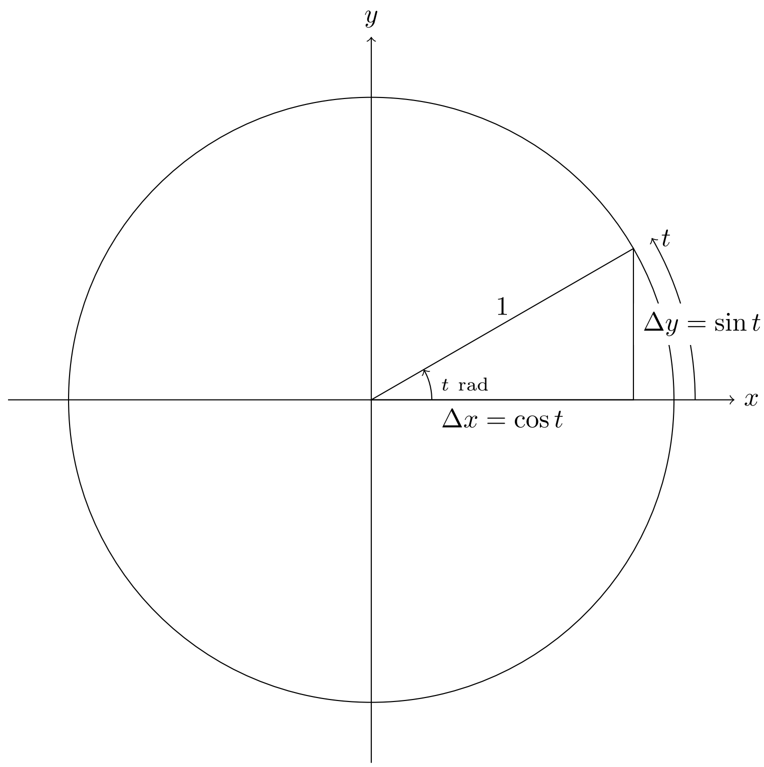
Suppose you have a slanted line (e.g., a road on a hillside). There are two ways to measure how "steep" it is. In algebra class, you have learned to measure its "steepness" as slope: vertical change divided by horizontal change, or "rise over run." On the other hand, in geometry class, the "steepness" would have been measured by looking at the angle between the slanted line and a horizontal line: the "angle of inclination." These two measures behave quite differently; for instance, lines of slopes 1 and 2 have quite different angles of inclinations, whereas lines of slopes 100 and 101 have very nearly the same angle of inclination (very slightly less than $\pi/2$, a right angle).

One question that bothered me when I was studying algebra and geometry was how to relate these two different measures of "steepness." As it happens, there is a way—using trigonometry.

Definition. The *tangent* of a real number t, written $\tan t$, is the slope of a line that makes an angle of t radians with the horizontal:



Like sine and cosine, tangent cannot be expressed in a (finite) combination of non-trigonometric operations. However, we can give an expression for tangent in terms of sine and cosine:



$$\tan t = \frac{\Delta y}{\Delta x} = \frac{\sin t}{\cos t}.$$

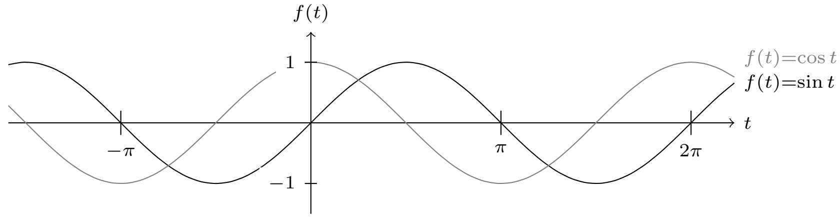
The equation

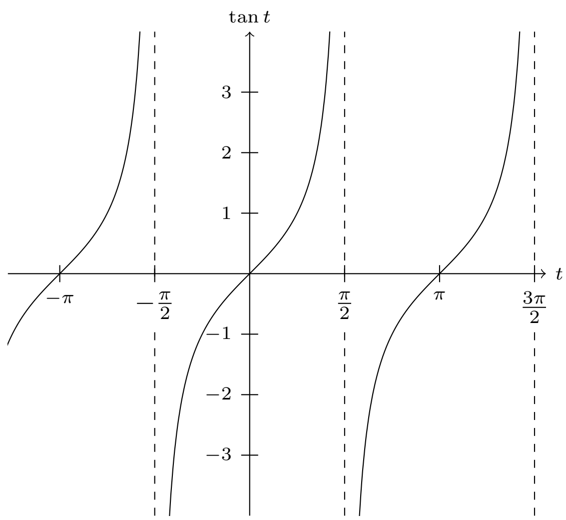
$$\tan t = \frac{\sin t}{\cos t}$$

is extremely important, and is often taken as the definition of the tangent function. There are stories that are sometimes used to help remember this equation. One such story concerns two sisters, named Sine and Cosine, who visit the beach one day and meet a Tanned Gent. They both want him, but in the end, the Tanned Gent chooses Sine over Cosine.

2 Periodicity

The graphs of the sine, cosine, and tangent are shown below:





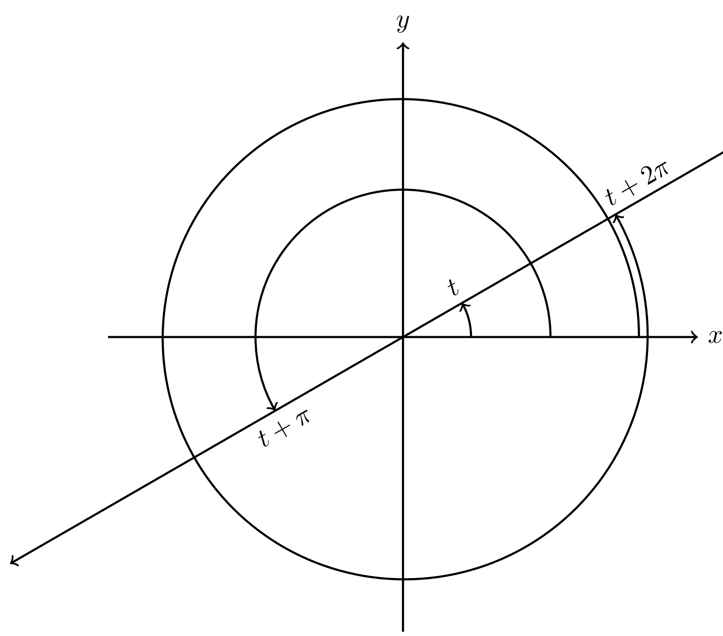
These functions exhibit a phenomenon called *periodicity*: if you translate any of the three graphs left or right by the correct amount (the *period*), then you get back the graph you started with.

Definition. Let f be a function (e.g., $f(x) = \sin(x)$ for all x). We say that a positive real number p is a *period* of f if

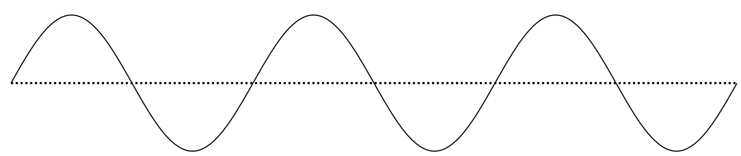
$$f(x) = f(x + c)$$

for all x in the domain of f. We say that f is *periodic* if it has a period. The smallest period p, if it exists, is called "the" period of f.

The graphs of sine, cosine, and tangent suggest that sine and cosine are periodic of period 2π, while tangent is periodic of period π. We can confirm this by thinking about the definitions: if you add 2π radians to an angle, you go all the way around the circle and get back the same angle, so of course it has the same sine and cosine. Similarly, if you go all the way around a circle, you get a line with the same slope, so 2π is *a* period of the tangent function; but it is not *the* period, since there is a smaller period (namely π). If you add π radians to an angle, you go halfway around the circle; you get a line with the same slope, i.e., this angle has the same tangent as the angle you started with.



3 Sinusoidal functions; amplitude



Many phenomena in nature are more or less periodic—for instance, seasons, waves, tides, pendula,.... If we want to provide formulas that allow us to approximate these things mathematically, then we need periodic functions; and the most generally useful periodic functions seem to be based on the sine and cosine functions. Essentially, any wave function like the one pictured above is called a *sinusoidal* function; and all sinusoidal functions can be described as geometric shifts of the sine function.

Let s be a sinusoidal function. Then s has the form

$$s(t) = A \sin \frac{t - t_0}{p/2\pi} + C,$$

where

t_0 controls where the wave starts,

p is the period,

A is the amplitude, and

C is the vertical translation.

This provides a geometric description as follows: Start with the sine function, and then

1. Shift it right by t_0;
2. Stretch it horizontally by $p/2\pi$;
3. Stretch it vertically by A; and finally
4. Shift it up by C.

Note, in particular, that the cosine function is a sinusoidal function, and may be described as

$$\cos t = \sin(t + \tfrac{\pi}{2});$$

the cosine function is obtained by shifting the sine function to the right by two.

4 Other trig functions

There are three other trigonometric functions, which are the reciprocals of the three functions already introduced. It's not entirely clear why these functions deserve names of their own, or why those names are what they they are, but I suppose you need to know them to understand what other people mean when they use them:

Definition. The *cosecant* of t, denoted $\csc t$, is the reciprocal of $\sin t$:

$$\csc t = \frac{1}{\sin t}.$$

The *secant* of t, denoted $\sec t$, is the reciprocal of $\cos t$:

$$\sec t = \frac{1}{\cos t}.$$

The *cotangent* of t, denoted $\cot t$, is defined by

$$\cot t = \frac{\cos t}{\sin t}.$$

The following equations also hold whenever they make sense:

$$\cot t = \frac{1}{\tan t} = \frac{\csc t}{\sec t}.$$

However, these cannot be used as the definition for the cotangent because of issues with division by zero. (E.g., $\tan t$, and hence $1/\tan t$, is undefined when $\cot t = 0$.)

5 Trigonometric identities

See pp. 47–48 in the textbook. These should be memorized.

Assignment 2 (due Monday, 9 January)

Section 0.7, Problems 1(d)–(f), 2(d)–(f), 16, and 17.

Section 1.4, Problems 1, 2, 15, and 16.

Assignment 3 (due Wednesday, 11 January)

Section 0.7, Problem 11. Parts (a) and (d) will be graded carefully.

Section 1.3, Problems 15 and 16. Problem 16 will be graded carefully.

Section 1.4, Problems 5 and 6. Problems 6 will be graded carefully.

Section 2.4, Problems 1 and 2. Problem 2 will be graded carefully.

Bonus Exercise. *Show that* sin *is not a rational function. (Hint: explain why* $\lim_{t\to\infty} f(t)$ *always exists (allowing* $\pm\infty$ *as the limit) if* f *is a rational function, but* $\lim_{t\to\infty} \sin t$ *does not exist. You do not need to give a rigorous proof.)*

Math 132, Lecture 3: Limits and derivatives of trigonometric functions

Charles Staats

Monday, 9 January 2012

1 Continuity of the trigonometric functions

The main result of this section is that the trigonometric functions are continuous wherever they are defined. In other words,

$$\begin{aligned}
\lim_{x\to x_0} \sin x &= \sin x_0 \\
\lim_{x\to x_0} \cos x &= \cos x_0 \\
\lim_{x\to x_0} \tan x &= \tan x_0,
\end{aligned}$$

where the last equation holds whenever $\tan x_0$ is defined. I won't bother to prove these; if this bothers you, see Section 1.4 in the textbook.

2 Derivative of sine

Let's see what happens when we try to find the derivative of $\sin x$ using the definition of the derivative:

$$\begin{aligned}
\frac{d}{dx}\sin x &= \lim_{h\to 0}\frac{\sin(x+h)-\sin x}{h} \\
&= \lim_{h\to 0}\frac{\sin x\cos h+\cos x\sin h-\sin x}{h} \\
&= \lim_{h\to 0}\frac{(\cos h-1)\sin x+\cos x\sin h}{h} \\
&= \lim_{h\to 0}\left(\frac{\cos h-1}{h}\cdot\sin x+\frac{\sin h}{h}\cdot\cos x\right) \\
&= \left(\lim_{h\to 0}\frac{\cos h-1}{h}\right)\sin x+\left(\lim_{h\to 0}\frac{\sin h}{h}\right)\cos x,
\end{aligned}$$

where the last equality is contingent on the existence of the limits in question. Thus, we can differentiate $\sin x$ if we can evaluate the two limits

$$\lim_{t\to 0}\frac{\cos t-1}{t} \qquad \text{and} \qquad \lim_{t\to 0}\frac{\sin t}{t}.$$

These are perhaps the first "interesting limits" as t approaches something finite that we've seen in this course. Unlike the limits we discussed last quarter, these limits cannot be resolved simply by canceling common factors from the numerator and denominator.

Here's a purely intuitive picture that should allow us to guess, and hopefully remember, what these limits are. Imagine zooming in, very close, to the circle at $t = 0$ radians; i.e., the point $(0, 1)$. If you zoom in close enough, then the circle becomes virtually indistinguishable from its (vertical) tangent line.

Thus, the distance gone around the circle, t, is virtually indistinguishable from the height $\sin t$. Likewise, the horizontal distance, $1 - \cos t$, is essentially zero. So, we have that

$$\sin t \approx t \qquad \text{and} \qquad 1 - \cos t \approx 0$$

for t very close to 0. From these, we might guess that

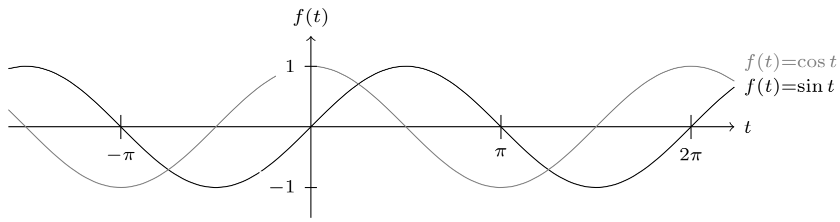
$$\lim_{t \to 0} \frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{t \to 0} \frac{1 - \cos t}{t} = 0;$$

and we would be correct.

Once we have these limits, our earlier work shows that

$$\begin{aligned} D_x \sin x &= \left(\lim_{h \to 0} \frac{\cos h - 1}{h} \right) \sin x + \left(\lim_{h \to 0} \frac{\sin h}{h} \right) \cos x \\ &= 0 \sin x + 1 \cos x \\ &= \cos x. \end{aligned}$$

If we look at the graphs of sine and cosine, this makes some sense:



The cosine function is positive precisely where the sine function is increasing; moreover, the peaks of the cosine wave are precisely the spots where the sine function is increasing fastest (i.e., has the greatest slope).

3 Other derivatives

We can use the definitions, plus the rules we know for differentiating, to find derivatives of the other trigonometric functions:

$$\begin{aligned}
\cos t &= \sin\left(\tfrac{\pi}{2} - t\right) \\
\frac{d}{dt}\cos t &= \cos\left(\tfrac{\pi}{2} - t\right) \cdot \frac{d}{dt}\left(\tfrac{\pi}{2} - t\right) && \text{(Chain Rule)} \\
&= (\sin t) \cdot (-1) \\
&= -\sin t
\end{aligned}$$

$$\begin{aligned}
\tan t &= \frac{\sin t}{\cos t} \\
\frac{d}{dt}\tan t &= \frac{(\cos t)(\cos t) - (\sin t)(-\sin t)}{\cos^2 t} && \text{(Quotient Rule)} \\
&= \frac{\cos^2 t + \sin^2 t}{\cos^2 t} \\
&= \frac{1}{\cos^2 t} && \text{(Pythagorean identity)} \\
&= \sec^2 t && \text{(Definition of secant function)}
\end{aligned}$$

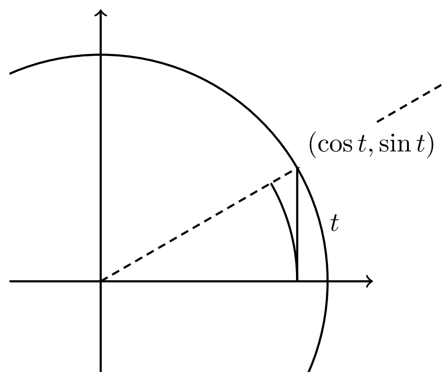
I'll leave the derivatives of secant, cosecant, and cotangent to you as an exercise. (Perhaps we'll do one of these in class, if there's time.)

4 Proofs of the limits

Now, let's give something approaching a rigorous justification for the two limits we discussed earlier from a purely intuitive point of view:

$$\lim_{t\to 0}\frac{\sin t}{t} = 1 \qquad \text{and} \qquad \lim_{t\to 0}\frac{1-\cos t}{t} = 0;$$

For the first one, the idea is to use the Squeeze Theorem (which I have also seen called the Sandwich Theorem): If we can show that, in a neighborhood of zero, $(\sin t)/t$ lies between two functions that both approach 1 as $t \to 0$, then necessarily, $(\sin t)/t$ also approaches 1 as $t \to 0$. To do this, we will sandwich $\sin t$ between two functions. Here's the picture:



The vertical line has height $\sin t$, the outer arc has length t, and the inner arc has length t times its radius, i.e., $t \cos t$. The length of the vertical line is sandwiched between the lengths of the two arcs (this can be made more obvious using areas). Thus, $\sin t$ is sandwiched between $t \cos t$ and t. In particular, for $t > 0$,

$$\begin{gathered} t \cos t < \sin t < t \\ \cos t < \frac{\sin t}{t} < 1 \\ \lim_{t \to 0^+} \cos t \leq \lim_{t \to 0^+} \frac{\sin t}{t} \leq \lim_{t \to 0^+} 1 \\ 1 \leq \lim_{t \to 0^+} \frac{\sin t}{t} \leq 1. \end{gathered}$$

The left-hand limit is similar, but with some of the inequalities going in the opposite direction.

Assignment 3 (due Wednesday, 11 January)

Section 0.7, Problem 11. Parts (a) and (d) will be graded carefully.

Section 1.3, Problems 15 and 16. Problem 16 will be graded carefully.

Section 1.4, Problems 5 and 6. Problems 6 will be graded carefully.

Section 2.4, Problems 1 and 2. Problem 2 will be graded carefully.

Bonus Exercise. *Show that* sin *is not a rational function. (Hint: explain why* $\lim_{t\to\infty} f(t)$ *always exists (allowing* $\pm\infty$ *as the limit) if* f *is a rational function, but* $\lim_{t\to\infty} \sin t$ *does not exist. You do not need to give a rigorous proof.)*

Assignment 4 (due Friday, 13 January)

Section 0.7, Problem 24.

Section 1.4, Problems 4 and 17. Problem 17 will be graded carefully.

Section 1.5, Problem 42. (Make sure you are looking at the limit as x approaches *infinity*. You may want to use the Squeeze Theorem.) This will be graded carefully.

Section 2.4, Problems 3 and 13. Problem 13 will be graded carefully, but problem 3 may be worth a second look on your part.

Section 2.5, Problem 9.

Math 132, Lecture 4: The Chain Rule with trigonometric functions; Review of Critical Points

Charles Staats

Wednesday, 11 January 2012

No class Monday

Monday, January 16 is Martin Luther King Day. We will not be having class.

1 Differentiating formulas that include trigonometric functions

There's technically nothing new in this section. We've already discussed the rules for differentiating sin, cos, tan, etc., and you already know the rules (product rule, chain rule, etc.) for differentiating more complicated formulas when you know how to differentiate the simpler functions of which they are composed. This section is going to consist of a few examples of how to combine these rules.

Example 1. (Example 4, p. 120 in the textbook) If $y = \sin 2x$, find $\frac{dy}{dx}$.

Solution 1: The long way. We may rewrite y as $y = \sin u$, where $u = 2x$. Then we have

$$\frac{dy}{du} = \cos u, \qquad \frac{du}{dx} = 2.$$

Hence, the Chain Rule gives us that

$$\begin{aligned} \frac{dy}{dx} &= \frac{dy}{du}\frac{du}{dx} \\ &= (\cos u) \cdot 2 \\ &= 2\cos(2x). \end{aligned}$$

□

Solution 2: The short way.

$$\begin{aligned}\frac{d}{dx}\sin 2x &= \cos(2x)\cdot\frac{d}{dx}(2x)\\ &= 2\cos 2x.\end{aligned}$$ □

What I call the "long way" of applying the Chain Rule involves actually specifying what u is in the equation

$$\frac{dy}{dx} = \frac{dy}{du}\frac{dy}{dx}.$$

When doing examples in class, I will typically use the "short way;" however, when grading, I will give full credit to either method used correctly.

Example 2. Find $\dfrac{d}{dt}\sin^2(t^3)$.

Solution.

$$\begin{aligned}\frac{d}{dt}\sin^2(t^3) &= \frac{d}{dt}\left(\sin t^3\right)^2\\ &= 2(\sin t^3)\cdot\frac{d}{dt}(\sin t^3)\\ &= 2\sin t^3\cdot\,\cos t^3\cdot\frac{d}{dt}t^3\\ &= 2\sin t^3\cdot\,\cos t^3\cdot 3t^2.\end{aligned}$$ □

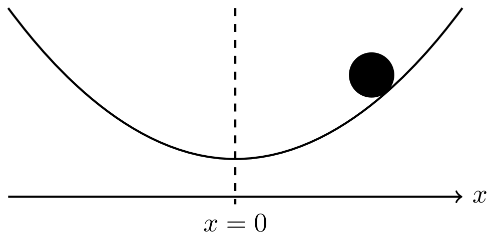
Example 3. Find $D_y(y\cos y^2)$.

Solution.

$$\begin{aligned}D_y y\cos y^2 &= (D_y y)\cos y^2 + y(D_y\cos y^2) && \text{(Product Rule)}\\ &= 1\cos y^2 + y\cdot(-\sin y^2)\cdot D_y(y^2)\\ &= \cos y^2 + y(-\sin y^2)(2y)\\ &= \cos y^2 - 2y^2\sin y^2.\end{aligned}$$ □

2 Why people really care about sinusoidal functions: The differential equation $x'' = -x$

Consider the situation of a ball in a bowl:



Consider the ball's horizontal position, x, as a function of time t. When the ball rolls to the right or left of the center point $x = 0$, it will experience a force pushing it back toward the center. Moreover, the farther away from the center it goes, the greater this force will be. Now it is a basic fact of physics[1] that force determines acceleration d^2x/dt^2. Thus, we may guess that this situation satisfies, at least approximately, the differential equation

$$\frac{d^2x}{dt^2} = -kx,$$

for some constant k. In other words, the bigger $|x|$ gets (i.e., the farther the ball gets from the center), the greater will be the ball's acceleration back towards the center. The negative sign indicates that this acceleration is in the opposite direction from x, i.e., back towards the center.

Now, note that the function $x(t) = \sin \sqrt{k}t$ satisfies

$$\begin{aligned} x(t) &= \sin \sqrt{k}t \\ x'(t) &= \sqrt{k} \cos \sqrt{k}t \\ x''(t) &= \sqrt{k} \cdot \sqrt{k} \cdot (-\sin \sqrt{k}t) \\ &= -k \sin \sqrt{k}t \\ &= -k\, x(t). \end{aligned}$$

In other words, this sinusoidal function is a solution of the differential equation. More generally, one can show that any function satisfying this differential equation is sinusoidal; and so the motion of the ball in a bowl is described, more or less[2], by a sinusoidal function.

Moreover, this sort of differential equation, in which some sort of force goes in the opposite direction from what it controls, shows up all over. Certainly it occurs in many other physics situations (e.g., springs

[1] Newton's Second Law, to be precise.

[2] The differential equation is only approximately true; a more exact version would require more information, including the exact shape of the bowl.

and pendulums, as well as the motion of air molecules in sound waves), but also in areas like economics and ecology. I have even seen it applied to romance: see `http://opinionator.blogs.nytimes.com/2009/05/26/guest-column-loves-me-loves-me-not-do-the-math/`.

3 Critical Points: A quick review

We will now briefly review the notion of critical points, which we discussed at the end of last term.

Definition. Let f be a function defined on a closed interval $[a, b]$. A *critical point* of f is a point c that is of at least one of the following three types:

(i) An endpoint: c is equal to either a or b.

(ii) A singular point: f is not differentiable at c, i.e., $f'(c)$ does not exist. This could mean that the graph of f has a corner at c, but it could also mean that the tangent line of f at c is vertical.

(iii) A stationary point: $f'(c) = 0$. These are probably the most interesting for us, but it is important to remember the other two. They are called stationary points for the following reason: when $f(t)$ describes the position of an object at time t, then $f'(t)$ is the velocity of that object at time t. Thus, the "stationary points" are precisely those points where the velocity is 0, i.e., the object is "not moving" or "stationary."

I really don't care if you remember the names "singular point" and "stationary point," but you do need to make sure that if I ask you to find the critical points of a function, you remember to look for all three kinds.

The reason we care about critical points is the following:

Theorem. Let f be a continuous function defined on a *closed* interval $[a, b]$. Then f attains a maximum and a minimum on this interval. Moreover, the only points at which f can attain a maximum or minimum are the critical points of f.

Thus, if someone gives us a function and we want to maximize or minimize it (say, we want to maximize profit), here's the basic strategy:

1. Find the critical points.
2. Evaluate the function at each of the critical points.
3. Take the maximum (minimum) of the resulting function values.

This is also one of the most important reasons for understanding the derivative: in order to find singular points and stationary points, we have to use the derivative.

Assignment 4 (due Friday, 13 January)

Section 0.7, Problem 24.

Section 1.4, Problems 4 and 17. Problem 17 will be graded carefully.

Section 1.5, Problem 42. (Make sure you are looking at the limit as x approaches *infinity*. You may want to use the Squeeze Theorem.) This will be graded carefully.

Section 2.4, Problems 3 and 13. Problem 13 will be graded carefully, but problem 3 may be worth a second look on your part.

Section 2.5, Problem 9.

Assignment 5 (due Wednesday, 18 January)

Section 2.5, Problems 10, 11, 12, and 33. Problems 10 and 12 will be graded carefully.

Section 2.6, Problems 5 and 6. Problem 6 will be graded carefully.

Section 2.7, Problem 11.

Section 3.1, Problems 7, 17, 18. Problems 7 and 18 will be graded carefully.

Math 132, Lecture 5: Min/Max problems; Monotonicity

Charles Staats

Friday, 13 January 2012

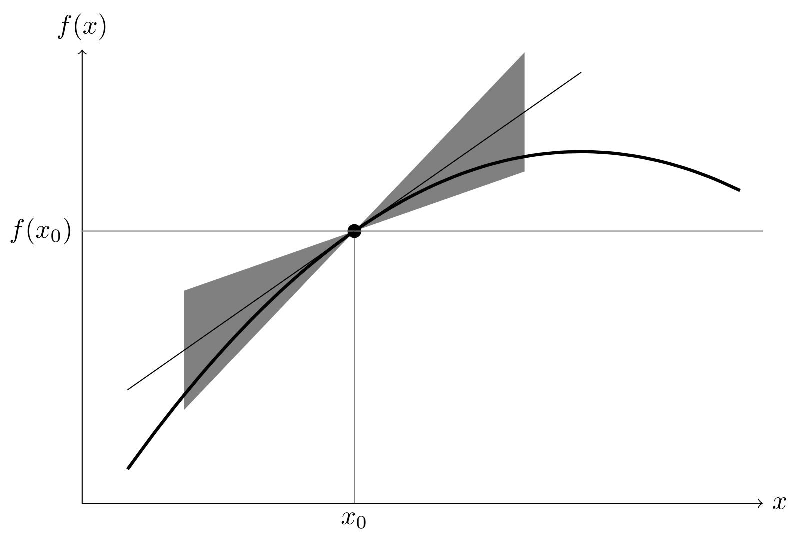
No class Monday

Monday, January 16 is Martin Luther King Day. We will not be having class.

1 Minima and Maxima: the theory

Theorem. Let f be a *continuous* function with domain a *closed* interval $[a, b]$. Then the only points where f could possibly equal its extreme values are the critical points.

Idea of proof. We prove the contrapositive. Suppose x_0 is not a critical point. We will show that $f(x_0)$ is not an extremal value of f.



Since x_0 is not a critical point, x_0 is differentiable and $f'(x_0) \neq 0$. In other words, f has a tangent line at x_0 that is not horizontal. Thus, for x sufficiently close to x_0, $f(x)$ is contained in a narrow cone about the tangent line.

Since the tangent line is not horizontal, if we make the cone sufficiently narrow, we can ensure that the values of f immediately to the right of x_0 (if the slope is positive) or immediately to the left of x_0 (if the slope is negative) are above $f(x_0)$. Since x_0 is not a critical point, it is not an endpoint of the domain, so f does have values immediately to the left and right of x_0. Hence, $f(x_0)$ is not an maximum of f.

Similar reasoning shows that $f(x_0)$ is not a minimum value of f. □

2 Maxima and minima: example

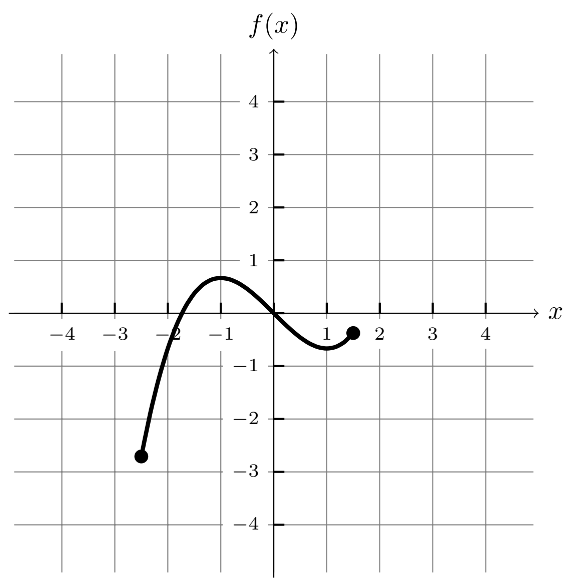
In other words, if we know f is a continuous function on $[a, b]$, then the following procedure will allow us to find the minima and maxima of f on $[a, b]$:

1. Find the critical points of f (all three kinds).
2. Evaluate f at each of the critical points.
3. The largest of the resulting values is the maximum value of f on $[a, b]$. The least of the resulting values is the minimum value of f on $[a, b]$.

Example 1. Find the critical points, minimum, and maximum for the function f given by

$$f(x) = \tfrac{1}{3}x^3 - x$$

on the closed interval $[-2.5, 1.5]$.



Solution to Example 1. First, we note that f is continuous (since it is a polynomial) and has domain equal to a closed interval, namely, $[-2.5, 1.5]$. Thus, we can follow the procedure:

1. *Find the critical points of f.* There are three kinds of critical points to consider:

   (a) The endpoints are -2.5 and 1.5.

   (b) The derivative of f is $f'(x) = x^2 - 1$. This is defined everywhere on the interval $[-2.5, 1.5]$. Thus, there are no singular points.

   (c) The stationary points are the zeros of the derivative, i.e., those points x such that

   $$\begin{aligned} x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

   Since both 1 and -1 lie within the interval $[-2.5, 1.5]$, they are both critical points.

   Thus, the critical points of f are -2.5, -1, 1, and 1.5.

2. *Evaluate f at each of the critical points.* We have

x	$f(x)$
-2.5	$-65/24$
-1	$2/3$
1	$-2/3$
1.5	$-3/8$

3. The greatest of the resulting values is $\frac{2}{3}$, since it is the only positive value of the four. It is therefore the maximum, and is attained at $x = -1$ and nowhere else.

   The least of the resulting values is $-\frac{65}{24}$, since it is the only value less than -1. It is therefore the minimum, and is attained at $x = -2.5$ and nowhere else. □

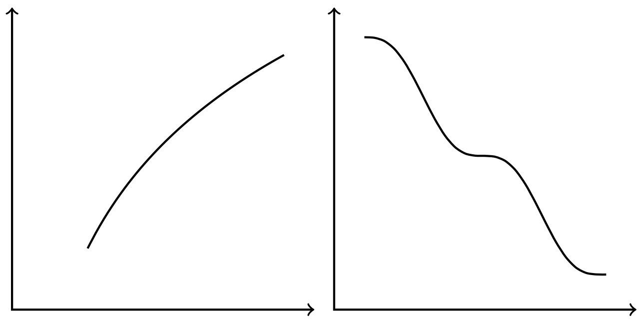


Figure 1: Strictly monotonic functions

3 Increasing and decreasing functions; monotonicity

For the study of increasing and decreasing functions, we no longer require our intervals to be closed.

Definition. (algebraic version) Let f be a function defined on an interval I. We say

(i) f is *increasing* on I if whenever $x_1 < x_2$ both lie in I, then $f(x_1) < f(x_2)$;

(ii) f is *decreasing* on I if whenever $x_1 < x_2$ both lie in I, then $f(x_1) > f(x_2)$.

Geometric version: we say

(i) f is *increasing* on I if every secant line over I has positive slope;

(ii) f is *decreasing* on I if every secant line over I has negative slope.

Exercise 2. Explain why the algebraic and geometric versions of "increasing" are really the same. Include a picture.

There's also a term that includes both concepts:

Definition. A function is *strictly monotonic* if it is increasing or decreasing.

For example, we might save ourselves some breath by saying that "if a function is strictly monotonic on a closed interval, then it attains its extreme values only at the endpoints." We've really made (at least) two statements at once: One statement about increasing functions, and a similar statement about decreasing functions.

Note that the definition above uses secant lines defined over *finite* length. We could also ask about the slope of the *tangent* lines—i.e., whether the function is "infinitesimally" increasing or decreasing. As it turns out, these two notions are closely related: "infinitesimally increasing" implies increasing, and the same for decreasing.

Theorem. (Monotonicity Theorem) Let f be continuous and differentiable on an interval I.

(i) If $f'(x)$ is positive for all x in I except possibly the endpoints, then f is increasing on I.

(ii) If $f'(x)$ is negative for all x in I except possibly the endpoints, then f is decreasing on I.

This theorem illustrates one of the principle themes of mathematics: relating an infinitesimal property (tangent lines have positive slope) to a more global property (function is increasing, i.e., secant lines have positive slope).

Note that the infinitesimal notion is *not* identical to the more general notion. For instance, the function

$$f(x) = x^3$$

has a horizontal tangent line at $(0, 0)$, so it is "infinitesimally constant" at this one (stationary) point. However, this function is increasing on the whole real line: even though one tangent line is horizontal, all the secant lines have positive slope.

Assignment 5 (due Wednesday, 18 January)

Section 2.5, Problems 10, 11, 12, and 33. Problems 10 and 12 will be graded carefully.

Section 2.6, Problems 5 and 6. Problem 6 will be graded carefully.

Section 2.7, Problem 11.

Section 3.1, Problems 7, 17, 18. Problems 7 and 18 will be graded carefully.

Assignment 6 (due Friday, 20 January)

Section 2.5, Problem 34. This will be graded carefully.

Section 2.6, Problems 13 and 14. Problem 14 will be graded carefully.

Section 2.7, Problem 12. This will be graded carefully.

Section 3.1, Problems 19, 20, and 29. Problem 20 will be graded carefully.

Section 3.2, Problems 1 and 2.

Lecture 5 notes, Section 3, Exercise 2 (p. 4). This will be graded carefully.

Bonus Exercise. *Forthcoming.*

Math 132, Lecture 6: Concavity

Charles Staats

Wednesday, 18 January 2012

1 A trigonometric limit problem

I noticed, when making out the first quiz, that doing certain kinds of trigonometric limits requires a certain trick. Because I have not taught this class before, I did not see this when I was assigning homework problems. I should have had a bunch of people pointing out to me that I had not explained how to do some of the homework problems. I should have been receiving emails, questions in class, and visits in office hours. Instead, no one said anything to me. And judging by the performance on the quiz, this was not because everyone already understood how to do this; it was because most people did not have the courage to point out that I had given them insufficient information. Please don't let this happen again.

[Side note: I do sometimes assign problems deliberately without explaining precisely how to do them. But even in these cases—for that matter, even if I think I *have* explained precisely how to do the problem—I welcome questions. I value the courage required to ask such questions far more than the talent to do the problems without asking.]

At any rate, I think it is time (overdue, in fact) for us to discuss this "trick." The key is to rewrite limits we don't know how to evaluate directly (like $\lim_{\theta \to 0} \frac{\sin 2\theta}{\theta}$) in terms of limits we do know (like $\lim_{\theta \to 0} \frac{\sin 2\theta}{2\theta} = 1$).

Example 1. Evaluate

$$\lim_{t\to 0} \frac{\sin 2t}{\sin 3t}.$$

Solution.

$$\begin{aligned}
\lim_{t\to 0} \frac{\sin 2t}{\sin 3t} &= \lim_{t\to 0} \frac{(\sin 2t)/t}{(\sin 3t)/t} \\
&= \lim_{t\to 0} \frac{(2\sin 2t)/(2t)}{(3\sin 3t)/(3t)} \\
&= \lim_{t\to 0} \frac{2}{3} \cdot \frac{(\sin 2t)/(2t)}{(\sin 3t)(3t)} \\
&= \frac{2}{3} \cdot \frac{\lim_{t\to 0} \frac{\sin 2t}{2t}}{\lim_{t\to 0} \frac{\sin 3t}{3t}} \\
&= \frac{2}{3} \cdot \frac{1}{1} \\
&= \frac{2}{3}.
\end{aligned}$$

□

2 Concavity

Let f be a function defined on a (not necessarily closed) interval I. Recall the geometric definition of increasing/decreasing:

Definition. (Geometric version) We say

(i) f is *increasing* on I if every secant line over I has positive slope;

(ii) f is *decreasing* on I if every secant line over I has negative slope.

There is a corresponding geometric version for the notion of *concavity*:

Definition. We say

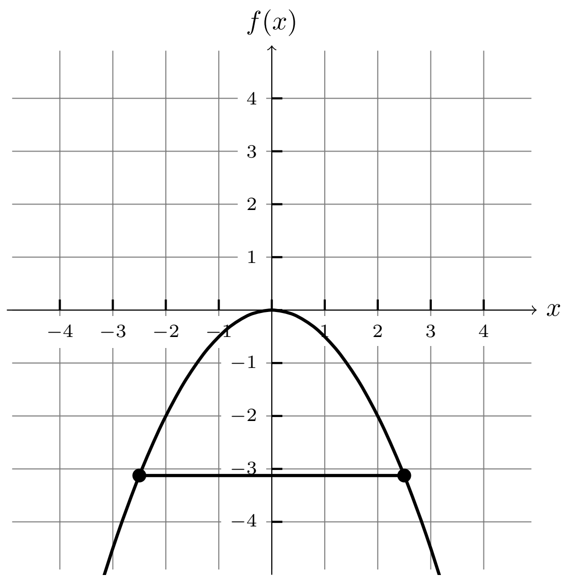
(i) f is *concave up* on I if every secant line over I lies strictly *above* the graph of f (except at its endpoints, which lie on the graph of f);

(ii) f is *concave down* on I if every secant line over I lies strictly *below* the graph of f (except at its endpoints, which lie on the graph of f).

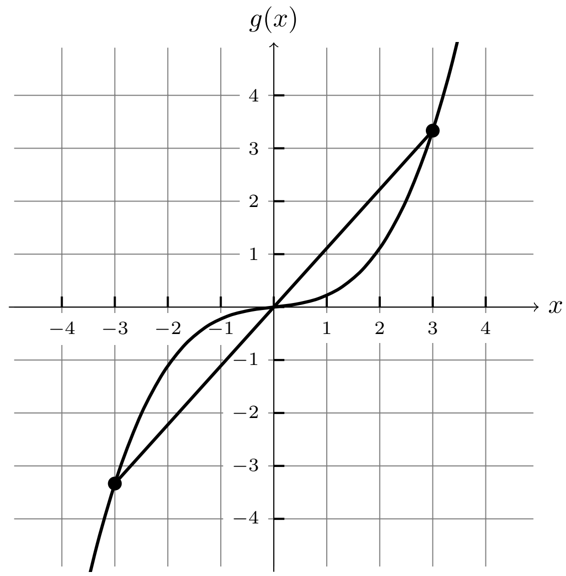
NOTE: This is *not* precisely equivalent to the definition given in the textbook. However, it is the definition that mathematicians actually care about.

Perhaps a couple of examples will make this clear:

Example 2. The function f defined by $f(x) = -\frac{1}{2}x^2$ is concave *down* on $(-\infty, \infty)$. Consequently, the secant line shown in the figure lies *below* the graph of f (except at its endpoints, which are on the graph of f). However, the secant line is horizontal (slope neither positive nor negative), which corresponds to the fact that f is neither increasing nor decreasing on $(-\infty, \infty)$.



Example 3. The function g defined by $g(x) = \frac{1}{9}(x^3 + x)$ is increasing on $(-\infty, \infty)$. Consequently, the secant line shown in the figure has positive slope. However, the secant line crosses the graph of g in the middle; it lies neither above the graph nor below the graph, which corresponds to the fact that f is neither concave up nor concave down on $(-\infty, \infty)$.



Like increasing/decreasing, the notion of concave up/down can be understood, more or less, by means of the derivative.

Theorem. Let f be a function defined on a not-necessarily-closed interval I. Suppose that at every point of I except possibly the endpoints, f is differentiable and the derivative is *increasing*. Then f is concave up.

Likewise, if the derivative is decreasing, then f is concave down.

This allows us to look at things in terms of the *second derivative:* Let f' be the derivative of f. Then the derivative of f' is f'', the second derivative of f. If f'' is positive on I, then f' is increasing on I, so the function f is concave up on I. Similarly, if f'' is negative on I, then f' is decreasing on I, so f is concave down on I.

The remainder of this lecture should be devoted to working out Examples 2 and 3 in terms of first and second derivatives. Unfortunately, I did not have time to type this up before the lecture, but I'll try to get it in before I post the notes online. In the mean time, there seems to be a large amount of space below each example, so you might use that space to take notes.

Assignment 6 (due Friday, 20 January)

Section 2.5, Problem 34. This will be graded carefully.

Section 2.6, Problems 13 and 14. Problem 14 will be graded carefully.

Section 2.7, Problem 12. This will be graded carefully.

Section 3.1, Problems 19, 20, and 29. Problem 20 will be graded carefully.

Section 3.2, Problems 1 and 2.

Lecture 5 notes, Section 3, Exercise 2 (p. 4). This will be graded carefully.

Assignment 7 (due Monday, 23 January)

Section 1.3, Problem 18. This will be graded carefully.

Section 1.4, Problems 7 and 8. Problem 8 will be graded carefully.

Section 2.5, Problems 15 and 16. Problem 16 will be graded carefully.

Section 3.1, Problems 8 and 9. Problem 9 will be graded carefully.

Section 3.2, Problems 19 and 20. Both of these will be graded carefully.

Bonus Exercise. *Suppose that a function f is increasing on the two closed intervals $[a, b]$ and $[b, c]$. Prove that f is increasing on the larger interval $[a, c]$. (Hint: use the algebraic definition of "increasing." You should not be using the derivative anywhere.)*

Test Friday, 27 January

The test will cover Assignments 1–7 and Lectures 1–7. The most relevant sections of the textbook are probably 0.7, 1.4, 2.4, 2.5, 3.1, and 3.2. Note also that there almost certainly will be a problem from Section 1.3, since many people showed an inability to handle this sort of problem on last quarter's final. I won't include a discussion of this in my lecture plans, but feel free to ask me to do an example.

Math 132, Lecture 7: Concavity; Local Extrema

Charles Staats

Wednesday, 18 January 2012

1 Concavity (for real, this time)

Let f be a function defined on a (not necessarily closed) interval I. Recall the geometric definition of increasing/decreasing:

Definition. (Geometric version) We say

(i) f is *increasing* on I if every secant line over I has positive slope;

(ii) f is *decreasing* on I if every secant line over I has negative slope.

There is a corresponding geometric version for the notion of *concavity*:

Definition. We say

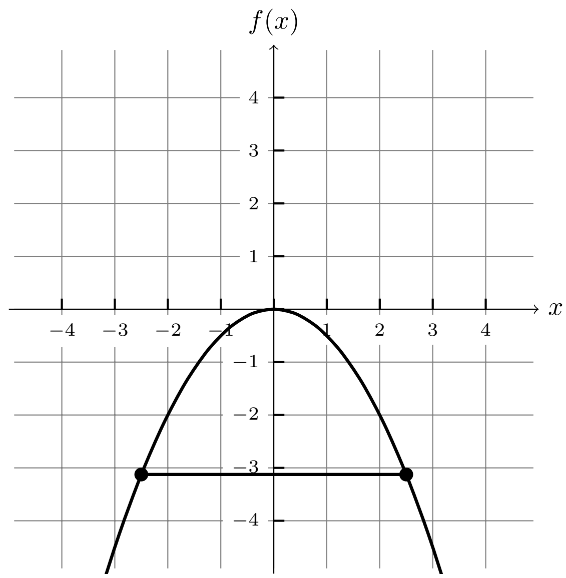
(i) f is *concave up* on I if every secant line over I lies strictly *above* the graph of f (except at its endpoints, which lie on the graph of f);

(ii) f is *concave down* on I if every secant line over I lies strictly *below* the graph of f (except at its endpoints, which lie on the graph of f).

NOTE: This is *not* precisely equivalent to the definition given in the textbook. However, it is the definition that mathematicians actually care about.

Perhaps a couple of examples will make this clear:

Example 1. The function f defined by $f(x) = -\frac{1}{2}x^2$ is concave *down* on $(-\infty, \infty)$. Consequently, the secant line shown in the figure lies *below* the graph of f (except at its endpoints, which are on the graph of f). However, the secant line is horizontal (slope neither positive nor negative), which corresponds to the fact that f is neither increasing nor decreasing on $(-\infty, \infty)$. It is, in fact, increasing on $(-\infty, 0]$ and decreasing[1]on $[0, \infty)$.



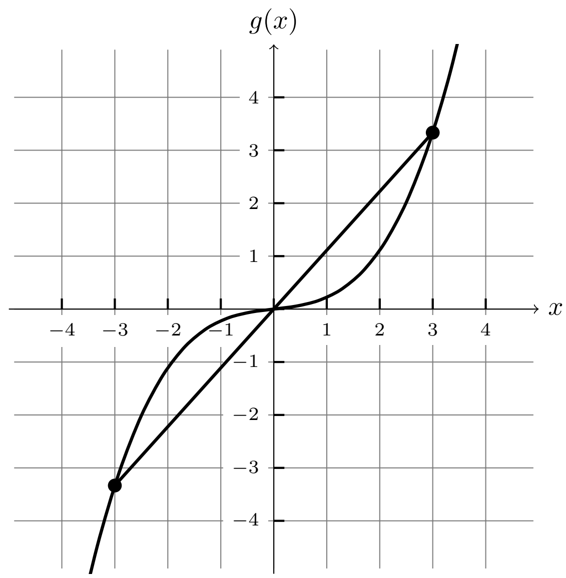
To actually prove that f is concave down, we can use the second derivative, as described below:

$$f(x) = -\tfrac{1}{2}x^2$$
$$f'(x) = -\tfrac{1}{2} \cdot 2x = -x$$
$$f''(x) = -1.$$

Since $f''(x) < 0$ on $(-\infty, \infty)$, f is concave down on $(-\infty, \infty)$.

[1]You may observe that both of this intervals include the point $x = 0$. This is an illustration of why we define "increasing" and "decreasing" *on an interval* but not *at a point*. If someone were to ask us whether f is increasing or decreasing (or neither) at $x = 0$, we'd have to say, "Since f is increasing on $(-\infty, 0]$, an interval that includes 0, f must be increasing at 0. Since f is decreasing on $[0, \infty)$, an interval that includes 0, f must be increasing at 0. So f is both increasing and decreasing at 0," a conclusion that is clearly bad. Thus, we should instead say, "It makes no sense to ask whether f is increasing or decreasing *at* 0; f can only be increasing or decreasing on an interval."

Example 2. The function g defined by $g(x) = \frac{1}{9}(x^3 + x)$ is increasing on $(-\infty, \infty)$. Consequently, the secant line shown in the figure has positive slope. However, the secant line crosses the graph of g in the middle; it lies neither above the graph nor below the graph, which corresponds to the fact that f is neither concave up nor concave down on $(-\infty, \infty)$.



We can analyze this example more carefully using the first and second derivative.

$$\begin{aligned} g(x) &= \tfrac{1}{9}(x^3 + x) \\ g'(x) &= \tfrac{1}{9}(3x^2 + 1) = \tfrac{1}{3}x^2 + \tfrac{1}{9} \\ g''(x) &= \tfrac{2}{3}x. \end{aligned}$$

Thus, we see that $g'(x) = \frac{1}{3}x^2 + \frac{1}{9} > 0$ for all $x \in (-\infty, \infty)$, so g is increasing on $(-\infty, \infty)$. However, $g''(x) = \frac{2}{3}x$ is not always positive or always negative. It is positive on $(0, \infty)$; this implies that g is concave up on $[0, \infty)$ (note that we do not care what g'' does at $x = 0$, since this is an endpoint). The second derivative g'' is negative on $(-\infty, 0)$; this implies that g is concave down on $(-\infty, 0]$.

Like increasing/decreasing, the notion of concave up/down can be understood, more or less, by means of the derivative.

Theorem. Let f be a function defined on a not-necessarily-closed interval I. Suppose that at every point of I except possibly the endpoints, f is differentiable and the derivative is *increasing*. Then f is concave up.

Likewise, if the derivative is decreasing, then f is concave down.

We won't prove this theorem until we get to the mean value theorem, but meanwhile, we can see how it is used. According to the theorem, if we have a differentiable function f that we want to show is concave up, we can look at whether the derivative f' is increasing. Write g for f'; we're trying to determine whether g is increasing. By our theorems from before, we can look at whether $g'(x)$ is *positive* for all x in the interval. Thus, our question about the concavity of f comes down to an inequality about $g' = f''$.

Theorem. Let f be a function defined on a not-necessarily-closed interval I. Suppose that at every point x of I except possibly the endpoints, $f''(x)$ is defined and $f''(x) > 0$. Then f is concave up on I.

Similarly, if $f''(x) < 0$, then f is concave down.

To illustrate this theorem, let's revisit our examples from before. (I've put the explanation together with the examples in the notes, so you should go back to them.)

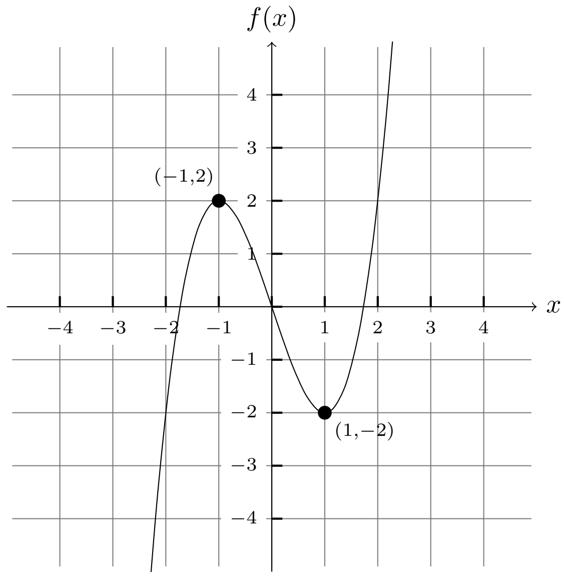
One final definition, that we don't really have time to go into right now:

Definition. (taken almost straight from the book) If f is continuous at c, we call c an *inflection point* of f if f is concave up on one side of c and concave down on the other side.

I *strongly* suggest you see Figure 17 on page 159 of the textbook; it is really a great illustration of different kinds of inflection points.

2 Local mins and maxes

Consider a function like $f(x) = x^3 - 3x$:



Note that $f'(x)$ is defined everywhere. If we want to look at this function on an open interval like $(-\infty, \infty)$, there are no endpoints to consider, so the only critical points we have to look at are the ones where $f'(x) = 0$.

$$\begin{aligned} f(x) &= x^3 - 3x \\ f'(x) &= 3x^2 - 3 \\ 3x^2 - 3 &= 0 \\ x^2 - 1 &= 0 \\ x^2 &= 1 \\ x &= \pm 1. \end{aligned}$$

Note that $f(-1) = (-1)^3 + 3 = 2$ and $f(1) = 1^3 - 3 = -2$. The two relevant points are plotted on the graph. Note that neither of them is a global maximum, since f attains arbitrarily large values (since $\lim_{x\to\infty} f(x) = \infty$; similarly, neither is a global minimum. But it seems like, in some sense, $(-1, 2)$ ought to be a *local* maximum: if x is forced to stay close to -1, then the point is a maximum. Likewise, $1, -2)$ ought to be a local minumum. Here are the definitions that make this precise:

Definition. Let f be a function, and c a point where f is defined. We say

(i) $f(c)$ is a *local maximum* of f if there exists an open interval (a, b) containing c, such that $f(c)$ is a maximum of f on[2] (a, b);

(ii) $f(c)$ is a *local minimum* of f if there exists an open interval (a, b) containing c such that $f(c)$ is a minimum of f on (a, b);

[2]Since f may not be defined on all of (a, b) (e.g., if c is an endpoint of the domain), we should technically say the following: $f(x) \le f(c)$ whenever x lies in (a, b) and $f(x)$ is defined.

(iii) $f(c)$ is a *local extremum* of f if it is a local minimum or a local maximum.

For instance, in the figure, $f(-1)$ is clearly not a maximum value on all of f. But it is a maximum value over $(-1.5, -.5)$, and therefore a local maximum.

Like global extrema, we can find local extrema using critical points. But I doubt I will have time to discuss this until next lecture.

Assignment 7 (due Monday, 23 January)

Section 1.3, Problem 18. This will be graded carefully.

Section 1.4, Problems 7 and 8. Problem 8 will be graded carefully.

Section 2.5, Problems 15 and 16. Problem 16 will be graded carefully.

Section 3.1, Problems 8 and 9. Problem 9 will be graded carefully.

Section 3.2, Problems 19 and 20. Both of these will be graded carefully.

Bonus Exercise. *Suppose that a function f is increasing on the two closed intervals $[a, b]$ and $[b, c]$. Prove that f is increasing on the larger interval $[a, c]$. (Hint: use the algebraic definition of "increasing." You should not be using the derivative anywhere.)*

Assignment 8 (due Wednesday, 25 January)

Section 3.2, Problems 21, 37, and 51. All three of these will be graded carefully.

Section 3.3, Problems 1 and 2. Problem 2 will be graded carefully.

Where is the function f defined by $f(x) = 2x + \cos x$ increasing? Where is it concave up? (Remember, your answer should be about *intervals*, not *points*.) Justify your answers using the first and second derivatives. This will be graded carefully.

Bonus Exercise. *Show that the function g defined by $g(x) = x + \sin x$ is increasing on all of $(-\infty, \infty)$. (Hint: the same ideas used in the Assignment 7 bonus exercise should be helpful in dealing with points where $f'(x) = 0$.)*

Test Friday, 27 January

The test will cover Assignments 1–7 and Lectures 1–(7, Section 1). The most relevant sections of the textbook are probably 0.7, 1.4, 2.4, 2.5, 3.1, and 3.2. Note also that there almost certainly will be a problem from Section 1.3, since many people showed an inability to handle this sort of problem on last quarter's final. I won't include a discussion of this in my lecture plans, but feel free to ask me to do an example.

Math 132, Lecture 8: More on local extrema; a practical example

Charles Staats

Monday, 23 January 2012

1 Criteria for local maxima and minima

These criteria can be used to determine whether a critical point is a local minimum, local maximum, or neither.

General criteria:

(a) If f is increasing to the immediate left[1] of x_0 and decreasing to the immediate right of x_0, then $f(x_0)$ is a local maximum.

(b) If f is decreasing to the immediate left of x_0 and increasing to the immediate right of x_0, then $f(x_0)$ is a local minimum.

(c) If f is decreasing on both sides of x_0 or increasing on both sides of x_0, then x_0 is not a local extremum.

First derivative test: Assume f is continuous in a neighborhood of x_0, and differentiable except possibly at x_0.

(a) If f' is positive to the immediate left[2] of x_0 and negative to the immediate right of x_0, then $f(x_0)$ is a local maximum.

(b) If f' is negative to the immediate left of x_0 and positive to the immediate right of x_0, then $f(x_0)$ is a local minimum.

(c) If the f' has the same sign to the immediate left of x_0 and to the immediate right of x_0, then $f(x_0)$ is neither a local min nor a local max.

Second derivative test: Assume f' and f'' both exist in a neighborhood of x_0, and $f'(x_0) = 0$ (i.e., x_0 is a "stationary point" of f).

(a) If $f''(x_0) < 0$, then $f(x_0)$ is a local maximum of f.

(b) If $f''(x_0) > 0$, then $f(x_0)$ is a local minimum of f.

[1] More precisely: if there exists some $a < x_0$ such that f is increasing on the interval $(a, x_0]$

[2] More precisely: if there exists some $a < x_0$ such that $f'(x) > 0$ for all x in the interval (a, x_0).

(c) We don't know anything this way if $f''(0) = 0$.

Example 1. (Examples 2 and 5, pages 163–165 in the book) Find the local extreme values of $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ on $(-\infty, \infty)$.

2 Practical Problems: an example

Example 2. (Example 1, p. 167 in the textbook) "A rectangular box is to be made from a piece of cardboard 23 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides, as in Figure 1 [omitted; I'll try to draw this on the board]. Find the dimensions of the box of maximum volume. What is this volume?

Assignment 8 (due Wednesday, 25 January)

Section 3.2, Problems 21, 37, and 51. All three of these will be graded carefully.

Section 3.3, Problems 1 and 2. Problem 2 will be graded carefully.

Where is the function f defined by $f(x) = 2x + \cos x$ increasing? Where is it concave up? (Remember, your answer should be about *intervals*, not *points*.) Justify your answers using the first and second derivatives. This will be graded carefully.

Bonus Exercise. *Show that the function g defined by $g(x) = x + \sin x$ is increasing on all of $(-\infty, \infty)$. (Hint: the same ideas used in the Assignment 7 bonus exercise should be helpful in dealing with points where $f'(x) = 0$.)*

Test Friday, 27 January

The test will cover Assignments 1–7 and Lectures 1–(7, Section 1). The most relevant sections of the textbook are probably 0.7, 1.4, 2.4, 2.5, 3.1, and 3.2. Note also that there almost certainly will be a problem from Section 1.3, since many people showed an inability to handle this sort of problem on last quarter's final. I won't include a discussion of this in my lecture plans, but feel free to ask me to do an example.

Assignment 9 (due Monday, 30 January)

Section 3.2, Problems 11, 12, and 22. Problems 12 and 22 will be graded carefully. These problems are intended as review, but they would also be good practice for the test Friday.

Section 3.3, Problems 3 and 4. Problem 4 will be graded carefully.

Section 3.4, Problem 9.

Math 132, Lecture 9:

Charles Staats

Wednesday, 25 January 2012

1 Example: local mins and maxes

Example 1. (Examples 2 and 5, pages 163–165 in the book) Find the local extreme values of $f(x) = \frac{1}{3}x^3 - x^2 - 3x + 4$ on $(-\infty, \infty)$. Do this in two different ways: using the First Derivative Test, and using the Second Derivative Test.

2 Practical Max/Min Problems

One of the things calculus is really good for is finding ways to maximize or minimize things within constraints. Typical questions might include things like, "I have a certain amount of fence. How do I enclose the largest possible area?" (a maximizing problem), or for a minimizing problem, "I need to enclose a certain area. How do I do it with the least possible area?"

Here's the example that will be used for illustrating the procedure:

Example 2. A farmer has 20 meters of fence and wants to build a rectangular enclosure. What is the greatest possible area he can enclose, and how can he enclose that area?

The procedure for handling these is something like the following:

1. *Convert the word problem into mathematics.* Often, this is the hardest part. Typically, you will have one quantity you care about (say, area) and one or more other quantities that control it (say, the length and width of a rectangular enclosure). You'll also have some additional information (in this case, that the total length of the fence is a particular amount, say 20). You need to record all this information in mathematical terms:
   (a) Identify the relevant quantities (in this case, length, width, and area) and name them (say, ℓ, w, and A, respectively).
   (b) Write down an equation that gives the quantity you want to max/minimize in terms of other quantites. In this case, we have
   $$A = \ell w.$$

(c) Write down any additional information in equations. In this case, the farmer has 20 meters of fence, so the perimeter of the rectangular enclosure should be 20. Thus, we have

$$2\ell + 2w = 20.$$

2. *Convert the mathematical problem into a calculus problem.* Play around with the equations until the quantity you care about is expressed in terms of a single other quantity. In our case, we have

$$\begin{aligned} 2\ell + 2w &= 20 \\ \ell + w &= 10 \\ \ell &= 10 - w \end{aligned}$$

Substituting in $10 - w$ for ℓ in the equation for A, we get

$$A = \ell w = (10 - w)w = 10w - w^2.$$

At this point, you might ask how you know which variable to solve for in terms of the others. Basically, it does not matter, as long as you find some variable that works. For this example, we could just as easily have taken $w = 10 - \ell$, and then gotten $A = 10\ell - \ell^2$; we would still get the same answer.

3. *Figure out what your domain is.* In our case, what values could reasonably be plugged in for w? Negative width makes no sense, so we have

$$w \geq 0.$$

Likewise, negative length makes no sense, so we have

$$\begin{aligned} \ell &\geq 0 \\ 10 - w &\geq 0 \\ 10 &\geq w. \end{aligned}$$

Thus, $0 \leq w \leq 10$, and so we are trying to maximize

$$A(w) = 10w - w^2$$

for w on the interval $[0, 10]$.

4. *Solve the calculus problem.* Differentiate the function you just found, find the critical points, etc. This is, often, the easiest part. It's also a demonstration of the "niceness" of calculus: the hardest part of the problem is all the algebra beforehand. Once we get it down to a calculus problem, it tends to be pretty easy.

   In our case, we have

$$\begin{aligned} A(w) &= 10w - w^2 \\ A'(w)&10 - 2w. \end{aligned}$$

   Thus, we have the following critical points:

(a) Endpoints: $w = 0$, $w = 10$.

(b) Points where $A'(w)$ is undefined: none.

(c) Points where $A'(w) = 0$:

$$\begin{aligned} 10 - 2w &= 0 \\ 10 &= 2w \\ 5 &= w. \end{aligned}$$

The critical points are 0, 5, and 10. We have

$$\begin{aligned} A(0) &= 10(0) - 0^2 = 0 \\ A(5) &= 10(5) - 5^2 = 50 - 25 = 25 \\ A(10) &= 10(10) - 10^2 = 100 - 100 = 0. \end{aligned}$$

Thus, A is maximized at $w = 5$, where it attains the value 25.

5. *Convert the answer back to practical terms.* The largest possible rectangular enclosure is 25 square meters; to achieve this, the farmer should make his rectangle with width 5 meters. The length is then given by

$$\ell = 10 - w = 10 - 5 = 5;$$

i.e., the length of the rectangle will also be 5 meters. In other words, the greatest area will be attained by a square enclosure.

Example 3. (Example 1, p. 167 in the textbook) "A rectangular box is to be made from a piece of cardboard 23 inches long and 9 inches wide by cutting out identical squares from the four corners and turning up the sides, as in Figure 1 [omitted; I'll try to draw this on the board]. Find the dimensions of the box of maximum volume. What is this volume?"

Test Friday, 27 January

The test will cover Assignments 1–7 and Lectures 1–(7, Section 1). The most relevant sections of the textbook are probably 0.7, 1.4, 2.4, 2.5, 3.1, and 3.2. Note also that there almost certainly will be a problem from Section 1.3, since many people showed an inability to handle this sort of problem on last quarter's final. I won't include a discussion of this in my lecture plans, but feel free to ask me to do an example.

Assignment 9 (due Monday, 30 January)

Section 3.2, Problems 11, 12, and 22. Problems 12 and 22 will be graded carefully. These problems are intended as review, but they would also be good practice for the test Friday.

Section 3.3, Problems 3 and 4. Problem 4 will be graded carefully.

Section 3.4, Problem 9.

Math 132, Lecture 10

Charles Staats

Monday, 30 January 2012

1 Practical Problems: Examples for finding the equation.

We've already discussed a fair amount how to find minima and maxima, once we've reduced ourselves to a calculus problem. Thus, I thought we should do a couple examples in which we *just* concentrate on finding the equation to be maximized or minimized.

Example 1. (Section 3.4, Problem 9) Find the volume of the largest open box that can be made from a piece of cardboard 24 inches square by cutting equal squares from the corners and turning up the sides.

Solution.

□

Example 2. (Section 3.4, Problem 19) A small island is 2 miles from the nearest point P on the straight shoreline of a large lake. If a woman on the island can row a boat 3 miles per hour and can walk 4 miles per hour, where should the boat be landed in order to arrive at a town 10 miles down the shore from P in the least time?

Solution.

□

2 Graphing functions with calculus

This is all about examples. Essentially, when you have the information we can obtain about a function using calculus, you can use this to help graph it.

Example 3. Let's start with an example from the test: Graph the function f defined by

$$f(x) = \cos(x + \tfrac{1}{2}\pi) - \tfrac{3}{2}x.$$

Note that I did *not* require you to graph this on the test.

Solution. First, let's simplify the formula a bit:

$$\begin{aligned} f(x) &= \cos(x + \tfrac{1}{2}\pi) - \tfrac{3}{2}x \\ &= \cos x \cos \tfrac{\pi}{2} - \sin x \sin \tfrac{\pi}{2} - \tfrac{3}{2}x \\ &= (\cos x)(0) - (\sin x)(1) - \tfrac{3}{2}x \\ &= -\sin x - \tfrac{3}{2}x. \end{aligned}$$

Now, we don't have the algebraic techniques necessary to solve this for 0 to find the x-intercepts. We can find the y-intercept:

$$f(0) = -\sin(0) - \tfrac{3}{2}(0) = 0 - 0 = 0.$$

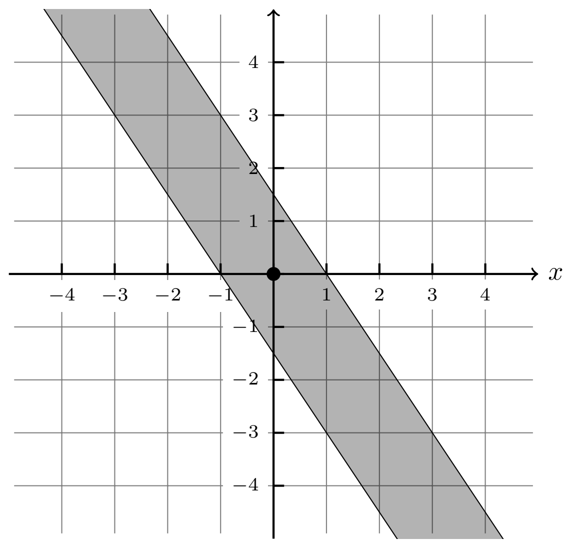
There's one more thing we can do pre-calculus, to prove a point stated in the textbook: "In graphing functions, there is not substitute for common sense." In this case, common sense tells us that perhaps we can use the inequality $-1 \le -\sin x \le 1$, which holds for all x. Subtracting $\frac{3}{2}x$ from both sides, we see that

$$-1 - \tfrac{3}{2}x \le -\sin x - \tfrac{3}{2}x \le 1 - \tfrac{3}{2}x,$$

i.e.,

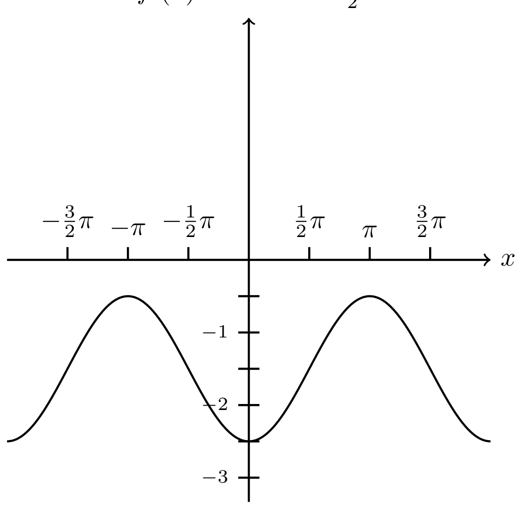
$$-1 - \tfrac{3}{2}x \le f(x) \le 1 - \tfrac{3}{2}x.$$

Graphically, this tells us that f must always lie between two parallel lines, as shown below. (I've also plotted the point $(0, 0)$, since we know $f(0) = 0$.)

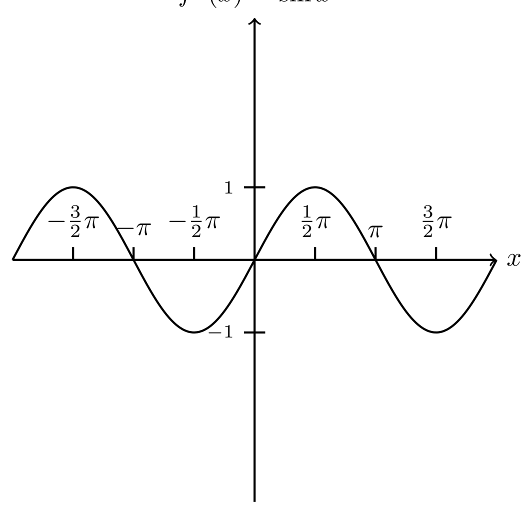


Now, let's get into the actual calculus. I've shown, below, f' and f'', together with their graphs:

$f'(x) = -\cos x - \frac{3}{2}$



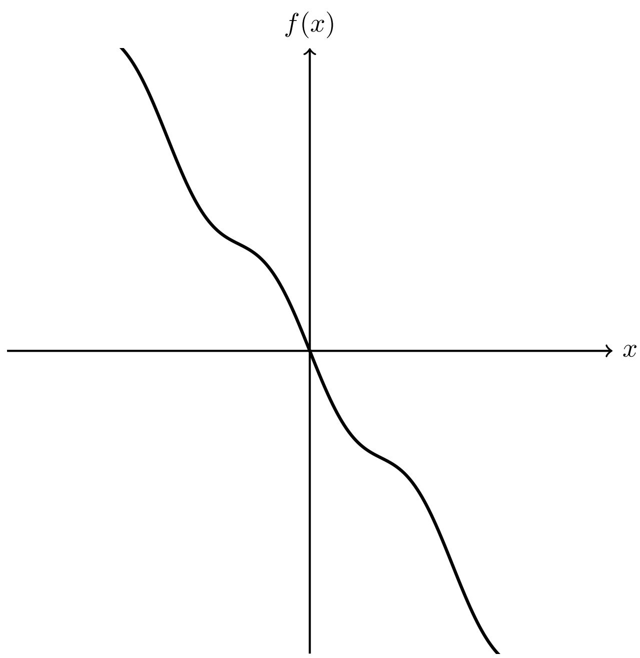
$f''(x) = \sin x$



From the graph of f', we can tell that $f'(x) < 0$ everywhere. Hence, f is decreasing on $(-\infty, \infty)$. (This also makes sense with what we found out about f lying between two downward-sloping parallel lines.) Furthermore, the tangent line never hits the horizontal, even for an instant. From the graph of f'', we can tell that f concave up on the intervals $[0 + 2\pi k, \pi + 2\pi k]$ (for each integer k) and concave down on the intervals $[\pi + 2\pi k, 2\pi + 2\pi k]$. The inflection points lie at the points

$$\begin{aligned}(\pi k, f(\pi k)) &= (\pi k, -\sin \pi k - \tfrac{3}{2}\pi k) \\ &= (\pi k, -\frac{3k\pi}{2}).\end{aligned}$$

We end up with



□

Assignment 10 (due Wednesday, 1 February)

Section 3.3, Problems 5, 6, 11, and 12. Problems 6 and 12 will be graded carefully.

Section 3.4, Problems 1, 2, 10, and 11. Problems 2 and 10 will be graded carefully.

Section 3.5, Problem 1. (You may need to skim Section 3.5 and/or ask your tutor to figure out how to do this.)

Assignment 11 (due Friday, 3 February)

Section 3.3, Problems 21 and 22. Both of these will be graded carefully.

Read Section 3.4, pp. 172–174, on economic applications. (Remember the three-pass method.)

Section 3.4, Problems 54–57. Given the interconnected nature of these problems, they will all be graded carefully.

Section 3.5, Problems 2–4. Problem 4 will be graded carefully.

Math 132, Lecture 11

Charles Staats

Wednesday, 1 February 2012

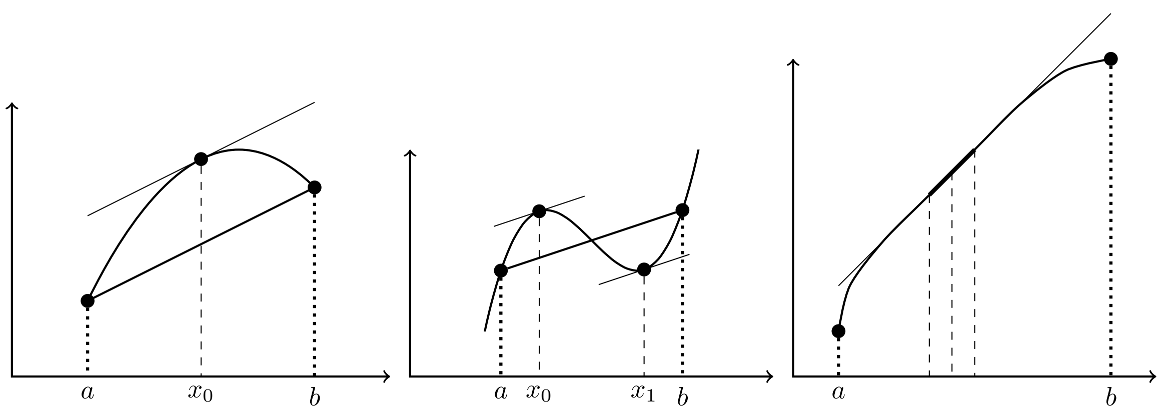
1 The Mean Value Theorem

The last two ideas we studied—solving practical problems and graphing functions with calculus—were mostly about examples. By contrast, the Mean Value Theorem is mostly about proofs. Here's the theorem:

Theorem. Let f be a function continuous on a closed interval $[a, b]$, such that f' is defined on the open interval (a, b). Let m be the slope of the secant line from $(a, f(a))$ to $(b, f(b))$. Then there exists at least one x_0 inside the open interval (a, b) such that

$$f'(x_0) = m.$$

Here's an illustration:



(a) one choice for x_0 (b) two choices for x_0 (c) infinitely many choices for x_0

A more velocity-based idea is given in this story (fictional, so far as I know): The distance between two tollbooths on a highway was 70 miles. The speed limit on this entire stretch was 55 miles per hour. One day, a cop at the second tollbooth was looking at drivers' toll records, and saw that one particular driver had covered the entire 70 miles in one hour. The cop gave the driver a ticket for going 15 mph over the speed limit.

The driver went to court, claiming that the ticket was unjustified because the cop had never witnessed him speeding. The cop replied that since the driver's average speed was 70 miles per hour, she knew by the Mean Value Theorem that the driver had, at some point, been going 70 miles per hour, even though she had not witnessed it.

2 Why mathematicians care about the Mean Value Theorem

The illustration of the cop and the driver is a nice view of the sorts of things one can do with the Mean Value Theorem: It is not so much for studying particular functions that we have formulas for, as for proving statements about functions when we have incomplete information. Maybe we don't know what the function was anywhere in the middle, if we know what it was at the beginning and at the end, then we know what the slope of the tangent line was somewhere.

The typical technique is proof by contradiction. Here's an example, proving a statement that made earlier without proof:

Theorem. Let f be a function continuous on $[a, b]$ and differentiable on (a, b). Suppose that $f'(x)$ is positive for all x in (a, b). Then f is increasing on $[a, b]$.

Proof. We assume the theorem is false, and show that this would contradict the Mean Value Theorem. Since the Mean Value Theorem is true, the theorem here must also be true.

The only way for the theorem to be false is if there is a function f, continuous on $[a, b]$ and differentiable on (a, b), such that $f'(x)$ is positive for all x in (a, b), but f is *not* increasing on $[a, b]$.

The only way for f not to be increasing on $[a, b]$ is if there exist x-values x_0 and x_1, both contained in $[a, b]$, such that $x_0 < x_1$ but the secant line between $(x_0, f(x_0))$ and $(x_1, f(x_1))$ does *not* have positive slope. Let $m \leq 0$ be the slope of this secant line. By the Mean Value Theorem, there exists an x-value c between x_0 and x_1 such that $f'(c) = m \leq 0$. But this is a contradiction, since by hypothesis, $f'(x)$ is positive for all x in (a, b). □

The basic idea of the proof is this: We know something about tangent lines (that their slope is always positive). We want to show the same thing holds for secant lines. To do this, we assume there is a secant line for which the slope is *not* positive, and use the Mean Value Theorem to produce a *tangent* line for which the slope is not positive.

Here's another example:

Theorem. Let f be a function continuous on $[a, b]$. Suppose that $f''(x)$ exists and is positive for all x in (a, b). Then f is concave up on $[a, b]$.

Assignment 11 (due Friday, 3 February)

Section 3.3, Problems 21 and 22. Both of these will be graded carefully.

Read Section 3.4, pp. 172–174, on economic applications. (Remember the three-pass method.)

Section 3.4, Problems 54–57. Given the interconnected nature of these problems, they will all be graded carefully.

Section 3.5, Problems 2–4. Problem 4 will be graded carefully.

Math 132, Lecture 12: Antiderivatives

Charles Staats

Friday, 3 February 2012

1 The velocity of a falling object

So far, we have been discussing how, when we are given a function, we can find its derivative. This has been a very useful thing; we can use the function's derivative to help us find important information about the function, such as its maxima and minima, and what its graph "looks like" in many ways (increasing/decreasing, concave up/down).

However, it is at least as important to be able to go in the opposite direction: given $f'(x)$, how do we find f? Or, to put it another way: if we have a function g and an equation of the form

$$\frac{dy}{dx} = g(x),$$

how do we find y as a function of x? This is the most basic type of *differential equation*: an equation that involves functions and their derivatives, rather than just numbers. Differential equations are incredibly important. Among other things, the laws of physics are practically all expressed as differential equations. And I imagine many of the "laws" of economics are as well.

For a specific example, let v be the velocity (in meters per second) of a "falling" object. (Remember that by *falling*, I mean that the object is not being held up by anything; but it may still be moving up, at least initially—for instance, if it was thrown up.) If we ignore air resistance, the laws of physics tell us that

$$\frac{dv}{dt} = -9.8;$$

in other words, the velocity is decreasing (or becoming more negative) steadily, at a rate of (9.8 meters per second) per second. This can help us figure out a formula for v. At the same time, it cannot be enough information to *determine* v, since this same law holds whether the object starts out moving up or down; and these motions have different v's.

Thus, we really have two questions:

1. What is *a* solution to the differential equation, and

2. What are *all* solutions?

In our case, the first question is quite easy to answer:

Solution. $v(t) = -9.8t$ is a solution, since for this v,

$$v'(t) = -9.8$$

for all t. □

However, this cannot be the *only* solution, because it only works for objects that start out with velocity 0. In real life, if I throw an object up, I should obtain a solution with $v(0)$ positive.

A bigger family of solutions may be found by adding constants:

Solution. For any constant C, let $v(t) = -9.8t + C$; then

$$v'(t) = -9.8,$$

so v is a solution to the differential equation. □

This seems like a much more reasonable candidate for "all possible solutions." If I throw the object up at a speed of v_0, then I can simply set

$$v(t) = -9.8t + v_0,$$

and I will have a solution such that the velocity at time 0 is $v(0) = v_0$, as it should be. And, as it turns out, these *are* all possible solutions.

2 Antiderivatives

Definition. Let f be a function. We say that another function F is an *antiderivative* for f if F is differentiable wherever f is defined, and $F'(x) = f(x)$.

The simplest antiderivatives to find are those given by reversing the power rule for differentiation:

Theorem. (Power Rule) Let $f(x) = x^n$. Then an antiderivative for f is given by

$$F(x) = \frac{1}{n+1}x^{n+1} + C,$$

where C can be any constant.

Proof.

$$F'(x) = \frac{1}{n+1} \cdot (n+1)x^n + 0 = x^n = f(x).$$

□

Note that what we did in the previous section to solve $dv/dt = -9.8$ was precisely an application of this rule.

Theorem. (linearity)

1. If F is an antiderivative of f and G is an antiderivative of g, then $F + G$ is an antiderivative of $f + g$.

2. Let a be a real number (constant). If F is an antiderivative of f, then aF is an antiderivative of af.

Theorem. (uniqueness up to constant) If F_1 and F_2 are both antiderivatives for f, then for some constant C,

$$F_1(x) = F_2(x) + C$$

for all x.

This means that, in some sense, "taking an antiderivative" is an operation we can apply to a function; the "answer" is unique up to constant.

Definition. For any function f, the *indefinite integral of* f, denoted

$$\int f(x)\, dx,$$

is the family of all antiderivatives of f. According to the theorem above, these all differ from each other by a constant (= vertical translation of the graph).

For example, the Power Rule may be restated as

$$\int x^n\, dx = \frac{1}{n+1} x^{n+1} + C.$$

The "$+C$" at the end is intended to indicate that we are considering a *family* of functions; different particular functions may be obtained by choosing different values for C.

Assignment 12 (due Monday, 6 February)

Complete the attached worksheet. Use a straightedge. All six of these will be graded carefully.

Section 3.3, Problems 13 and 14. Problem 14 will be graded carefully.

Section 3.4, Problems 19–21. Problem 19 will be graded carefully.

Section 3.5, Problem 28. This will be graded carefully.

Section 3.8, Problems 1 and 2.

Assignment 13 (due Wednesday, 8 February)

Section 3.4, Problem 12. This will be graded carefully.

Section 3.5, Problems 13, 14, and 31. Problems 14 and 31 will be graded carefully.

Section 3.6, Problems 22 and 27. Both of these will be graded carefully.

Section 3.8, Problems 3–6. Problems 4 and 6 will be graded carefully.

Math 132, Lecture 13: Indefinite integrals

Charles Staats

Monday, 6 February 2012

1 Theory

Recall, from last time, the definition of the indefinite integral:

Definition. For any function f, the *indefinite integral of* f, denoted

$$\int f(x)\,dx,$$

is the family of all antiderivatives of f; that is, the family of all functions F such that $F'(x) = f(x)$ for all x at which $f(x)$ is defined.

A note on the notation: the integral sign $\int$ was originally an elongated S. We will understand later why Leibniz (and most mathematicians since) thought that this was an appropriate notation.

Remark. We will later in this lecture need to understand why

$$\frac{d}{dx}\int f(x)\,dx = f(x).$$

This is, in essence, just a fancy notational way of saying that "the derivative of an antiderivative of f" is precisely f. In other words, the definition of antiderivative.

When we know a single antiderivative F for f, we typically write

$$\int f(x)\,dx = F(x) + C,$$

where C is taken to be a constant. If F is an antiderivative for f, then so is $F(x) + C$:

$$\frac{d}{dx}(F(x) + C) = F'(x) + 0 = f(x).$$

The theorem below tells us that, in fact, once we have a single antiderivative, every other antiderivative may be obtained from it by adding a constant:

Theorem. (uniqueness up to a constant) If F and F_1 are both antiderivatives for f, then

$$F_1(x) = F(x) + C$$

for some constant C.

Proof. A common technique in mathematics, when you are trying to prove a "hard" result, is to first prove an "easier" special case, and then show that the "hard" statement follows from the "easier" statement. In our case, we will consider the special case in which $f(x) = 0$ for all x. Clearly, $F(x) = 0$ is an antiderivative of f. Thus, in this case, the theorem says precisely the following:

Claim. If $F_1(x)$ is an antiderivative of 0, then

$$F_1(x) = 0 + C = C$$

for some constant C.

Proof. We are given that $F_1(x)$ is an antiderivative of 0, i.e., that

$$F_1'(x) = 0$$

for all x. We want to find a constant C such that $F_1(x) = C$ for all x. Let x_0 be any point in the domain of F_1, and set $C = f(x_0)$. Now we have a "candidate" value for C; we need to show that this value for C actually does what we want, i.e., that $F_1(x) = C$ for *all* x. To do this, we use the Mean Value Theorem.[1]

[Examine Figure 1 to help understand the following paragraph.] Let x be any point in the domain of F_1. We want to show that $F_1(x) = C$. Consider the secant line between x_0 and x. By the Mean Value Theorem, this secant line has the same slope as the tangent line to F_1 at some point x_1 between x_0 and x. Since $F_1'(x_1) = 0$, this tangent line is horizontal. Hence, the secant line is also horizontal. In other words,

$$\begin{aligned} F_1(x) &= F_1(x_0) \\ &= C, \end{aligned}$$

which is precisely what we wanted to show. ☐

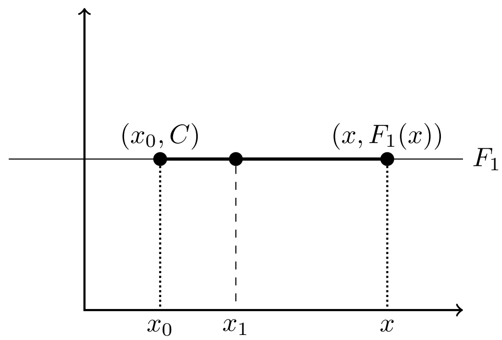


Figure 1: Since the secant line between x_0 and x is horizontal, $F_1(x) = C$.

Now, let's see how the whole theorem can be reduced to the special case we've just proved. We know that F and F_1 are both antiderivatives of f. In other words, for all x, $F'(x) = F_1'(x) = f(x)$. Let G be the function defined by

$$G(x) = F_1(x) - F(x).$$

Then

$$\begin{aligned} G'(x) &= F_1'(x) - F'(x) \\ &= f(x) - f(x) \\ &= 0. \end{aligned}$$

Thus, G is an antiderivative of 0. By the Claim, G is a constant function; in other words, there is a constant C such that

$$G(x) = C$$

for all x. Thus, by definition of G, we have

$$\begin{aligned} F_1(x) - F(x) &= G(x) = C \\ F_1(x) &= F(x) + C, \end{aligned}$$

which is exactly what we wanted to show. □

The "proof" part of this lecture is now concluded; for the rest of the lecture, we will focus on how to use what we now know to actually compute antiderivatives.

[1] I'm cheating a little bit here. To do this, I really need to know that the set over which $F_1'(x)$ is defined is an interval. And in fact, if it is not, the theorem may not hold. For instance, the piecewise-defined function

$$f(x) = \begin{cases} 0 & \text{if } x < 0, \\ 1 & \text{if } x > 0, \end{cases}$$

is an antiderivative for $f(x) = 0$ wherever it is defined (i.e., wherever $x \neq 0$), but it is not equal to a single constant.

2 Polynomials; sine and cosine

Recall, from last time, the power rule and linearity, which together allow us to integrate polynomials:

Theorem. (Power Rule)

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1} + C.$$

for any rational number n other than $n = -1$.

Theorem. (Linearity)

(i) $\int k f(x) \, dx = k \int f(x) \, dx$ for any nonzero constant k.

(ii) $\int [f(x) + g(x)] \, dx = \int f(x) \, dx \; + \; \int g(x) \, dx.$

Example 1. (Example 4a on p. 200 in the textbook)

$$\int (3x^2 + 4x) \, dx$$

Solution.

$$\begin{aligned} \int (3x^2 + 4x) \, dx &= 3 \int x^2 \, dx \; + \; 4 \int x \, dx \\ &= 3 \left(\tfrac{1}{3} x^3\right) + 4 \left(\tfrac{1}{2} x^2\right) + C \\ &= x^3 + 2x^2 + C. \end{aligned}$$

□

The rules for sine and cosine are not given in the textbook, but they are really quite simple:

Theorem. (integrating sine and cosine)

$$\int \cos x \, dx = \sin x + C \qquad \int \sin x \, dx = -\cos x + C.$$

Proof. Since

$$\frac{d}{dx} \sin x = \cos x,$$

we know that $\sin x$ is an antiderivative of $\cos x$. Thus, every antiderivative of $\cos x$ is of the form $\sin x + C$, for some constant C.

Since

$$\frac{d}{dx}(-\cos x) = -\frac{d}{dx} \cos x = -(-\sin x) = \sin x,$$

we know that $-\cos x$ is an antiderivative of $\sin x$. Thus, every antiderivative of $\sin x$ is of the form $-\cos x + C$, for some constant C. □

3 u-substitution

Recall that there are two differentiation rules with a fair amount of subtlety: the product rule, and the chain rule. The corresponding integration techniques are also fairly subtle, and (unlike, e.g., the power rule) have different names. The product rule for differentiation corresponds to "integration by parts," which we will study later (if ever). However, the chain rule corresponds to "u-substitution", which we will introduce now.

Theorem. (u-substitution)

Leibniz notation: If y can be given in terms of u and u can be given in terms of x, then

$$\int y \cdot \frac{du}{dx}\, dx = \int y\, du.$$

Functional notation:

$$\int f(g(x)) \cdot g'(x)\, dx = \int f(u)\, du.$$

Example 2. (Example 5a, p. 201 in the textbook) Evaluate

$$\int (x^4 + 3x)^{30}(4x^3 + 3)\, dx.$$

Solution. Let $u = x^4 + 3x$. The rest will be an in-class exercise. (Hint: first, find du, in terms of x and dx.)

□

Example 3. (Example 5b, p. 201 in the textbook) Evaluate

$$\int \sin^{10} x \cos x \, dx.$$

Solution. Let $u = \sin x$. Again, the rest will be an in-class exercise. □

Example 4. Evaluate $\int \sin(2x) \, dx$.

Solution. Let $u = 2x$. □

Proof of u-substitution. Leibniz notation: By the Chain Rule, we have

$$\begin{aligned} \frac{du}{dx}\,dx &= du \\ y\frac{du}{dx}\,dx &= y\,du \\ \int y\frac{du}{dx}\,dx &= \int y\,du. \end{aligned}$$

Functional notation: Let $u = g(x)$. By the Chain Rule,

$$\begin{aligned} \frac{d}{dx}\int f(u)\,du &= \frac{du}{dx}\cdot\frac{d}{du}\int f(u)\,du \\ &= \frac{du}{dx}\cdot f(u), && \text{by the remark back on the first page} \\ &= g'(x)\cdot f(g(x)), && \text{since } u = g(x). \end{aligned}$$

Thus, $\int f(u)\,du$ is a (family of) antiderivatives for $f(g(x))\cdot g'(x)$; or, in integral notation,

$$\int f(g(x))g'(x)\,dx = \int f(u)\,du.$$

□

Assignment 13 (due Wednesday, 8 February)

Section 3.4, Problem 12. This will be graded carefully.

Section 3.5, Problems 13, 14, and 31. Problems 14 and 31 will be graded carefully.

Section 3.6, Problems 22 and 27. Both of these will be graded carefully.

Section 3.8, Problems 3–6. Problems 4 and 6 will be graded carefully.

Assignment 14 (due Monday, 13 February)

Section 3.4, Problem 13. This will be graded carefully.

Section 3.5, Problems 15 and 32. Problem 32 will be graded carefully.

Section 3.6, Problem 27. This will be graded carefully.

Section 3.8, Problems 19, 20, 27, and 28. Problems 20 and 28 will be graded carefully.

Explain why the two statements of u-substitution, on p. 5 of this lecture, mean the same thing.

Math 132, Lecture 14: Sums and their limits

Charles Staats

Wednesday, 8 February 2012

1 No class Friday

There will be no class on Friday, February 10. Thus, Assignment 14 will instead be due on Monday, 13 February.

However, tutorial will proceed as usual on Thursday (tomorrow). There will even be a graded quiz.

2 Sums and Σ (sigma) notation

I assume that all of you know how to do a sum like

$$1 + 2 + 3 + \cdots + 10,$$

given enough time. We are going to study, instead, how to do sums like

$$1 + 2 + 3 + \cdots + n,$$

even when *you do not know* n. We will be able to find a "nice" formula for a sum like this in terms of n. This will be important shortly, since we will compute areas as limits of these sums as $n \to \infty$. But you don't need to worry about that just yet.

First of all, a few important formulas:

$$1 + 2 + 3 + \cdots + n = \tfrac{1}{2}n(n+1) \tag{1}$$

$$1^2 + 2^2 + 3^2 + \cdots + n^2 = \tfrac{1}{6}n(n+1)(2n+1) \tag{2}$$

$$1^3 + 2^3 + 3^3 + \cdots + n^3 = \left[\tfrac{1}{2}n(n+1)\right]^2. \tag{3}$$

The first formula can be remembered as "n times the average of 1 and n." (You might think of this as "the number of numbers" times "the average of the first number and the last number," but this is dangerous in that it fails for the second and third formulas.) The third formula is the square of the first. I don't know of any easy way to remember the second formula, although some of you might find $\frac{1}{3}(n+0)(n+\frac{1}{2})(n+1)$ an easier form to remember than the one given above. These formulas are important in order to be able to do other stuff; however, they are not particularly memorable (although I have tried to make them as memorable as I could). Thus, if you might need them on a test, I will give them to you.

We can use these equations to find formulas for more complex sums. However, we're going to need a better notation. For a demonstration, let's see try the following sum:

Example 1. (Example 3, p. 4 in the textbook) Find a formula, in terms of n, for

$$3(-4) + 4(-3) + 5(-2) + 6(-1) + \cdots + (n+2)(n-5).$$

You may have no idea how to start on this. That's not surprising. Seeing the problem written this way, I find it a bit intimidating myself. However, once we introduce sigma notation for sums, the solution will fall out much more easily.

Here's how sigma notation works: if you have an expression for the i^{th} term of the sum, say $\text{stuff}(i)$, then for the sum of n terms $\text{stuff}(1) + \text{stuff}(2) + \cdots + \text{stuff}(n)$, you write

$$\sum_{i=1}^{n} \text{stuff}(i).$$

Thus, for instance, we could write

$$1 + 2 + \cdots + n = \sum_{i=1}^{n} i$$
$$1^2 + 2^2 + \cdots + n^2 = \sum_{i=1}^{n} i^2$$
$$3(-4) + 4(-3) + 5(-2) + 6(-1) + \cdots + (n+2)(n-5) = \sum_{i=1}^{n} (i+2)(i-5).$$

The Greek capital letter Σ (sigma) is, as I understand it, the Greek version of the Roman letter S; it stands for "Sum." The Σ "operator" has some nice properties that the book, as usual, calls "linearity":

Theorem. ("linearity of Σ")

(i) $\sum_{i=1}^{n} c\,\text{stuff}(i) = c \sum_{i=1}^{n} \text{stuff}(i)$. This is just a fancy way of writing the distributive property.

(ii) $\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$. This is a fancy way of writing that it does not matter what order you add things in.

(iii) $\sum_{i=1}^{n}(a_i - b_i) = \sum_{i=1}^{n} a_i \; - \; \sum_{i=1}^{n} b_i$. To see this, we write

$$\begin{aligned}\sum_{i=1}^{n}(a_i - b_i) &= \sum_{i=1}^{n}(a_i + (-1)b_i) \\ &= \sum_{i=1}^{n} a_i + \sum_{i=1}^{n}(-b_i) \\ &= \sum_{i=1}^{n} a_i + (-1)\sum_{i=1}^{n} b_i \\ &= \sum_{i=1}^{n} a_i - \sum_{i=1}^{n} b_i.\end{aligned}$$

There's one more thing I should state: for any fixed c,

$$\sum_{i=1}^{n} c = \underbrace{c + c + \cdots + c}_{n \text{ times}} = nc.$$

Now, using this notation, together with the previously given formulas, we will return to the example at hand:

$$\begin{aligned}\sum_{i=1}^{n}(i+2)(i-5) &= \sum_{i=1}^{n} i^2 - 3i - 10 \\ &= \sum_{i=1}^{n} i^2 - \sum_{i=1}^{n} 3i - \sum_{i=1}^{n} 10 \\ &= \sum_{i=1}^{n} i^2 - 3\sum_{i=1}^{n} i - \sum_{i=1}^{n} 10 \\ &= \tfrac{1}{6}n(n+1)(2n+1) - 3 \cdot \tfrac{1}{2}n(n+1) - 10n.\end{aligned}$$

The last step is where we apply the formulas. Obviously, this can be simplified, and for many purposes, probably should be; but I'll leave it for now.

3 Areas as limits of infinite sums

Assignment 14 (due Monday, 13 February)

Section 3.4, Problem 13. This will be graded carefully.

Section 3.5, Problems 15 and 32. Problem 32 will be graded carefully.

Section 3.6, Problem 27. This will be graded carefully. (Note: this is a repeat problem.)

Section 3.8, Problems 19, 20, 27, and 28. Problems 20 and 28 will be graded carefully.

Explain why the two statements of u-substitution, on p. 5 of Lecture 13, mean the same thing.

Assignment 15 (due Wednesday, 15 February)

Section 3.4, Problem 59. This will be graded carefully.

Section 3.6, Problem 28. This will be graded carefully.

Section 3.8, Problems 21, 22, 29, and 35. Problems 22 and 35 will be graded carefully.

Section 4.1, Problems 38, 53, and 54. Problems 38 and 54 will be graded carefully.

Math 132, Lecture 15: Definite integrals

Charles Staats

Monday, 13 February 2012

1 Theoretical framework

We'd like to be able to just sit down and talk about "the area under a function $f(x)$," and in fact, this is what the earliest practitioners of calculus did. However, as the rigorous foundation of mathematics evolved—as mathematicians built the "skyscraper" of analysis to hold up the "cloud castle" of calculus—they ran into a problem: for some incredibly badly behaved functions, *the term "area underneath the function" simply makes no sense.* Asking what is the area under such a function is like asking what is $\lim x \to \infty \sin x$; the question simply has no answer.

In order to understand what is going on, mathematicians came up with the following procedure for dealing with a function f:

- Start with an intuitive idea of what the "area under f" should mean.
- Using this intuition, come up with a precise definition for a "lower area" $\underline{A}$ and an "upper area" $\overline{A}$. These should be approximations for the "true area" A such that, according to our intuition about the "true area," we will necessarily have
$$\underline{A} \leq A \leq \overline{A}.$$
- If it happens that $\underline{A} = \overline{A}$, then we know exactly what A must be. In this case, we say that the function is *integrable* (a fancy word for "it makes sense to talk about the area under the function"), and we define the *integral* to be A.
- If $\underline{A}$ is *not* equal to $\overline{A}$, then we can do one of two things:
  - give up, or
  - find a better procedure for getting $\underline{A}$ and $\overline{A}$.

The procedure we will introduce for finding $\underline{A}$ and $\overline{A}$ is called "Riemann integration." It's not the "best" procedure out there, but it will be more than adequate for our purposes.

2 Review: Finding the area under x^2 as a limit

Consider the following problem:

Example 1. What is the area of the region under the curve $y = x^2$ from $x = 0$ to $x = 1$? (See the picture.)

We more or less did this example in class last time (albeit without finishing), so I will just briefly give an idea of how we did it.

We don't (yet) have any simple, geometric formulas for the area under a parabola. But we can calculate the area of a bunch of little rectangles. So, let's approximate the area as a bunch of little rectangles:

Looking at it like this, we can even give a formula for our "approximate area" in terms of the number of rectangles: Let n be the number of rectangles. The area of the rectangle at x_i is

$$\text{width} \cdot \text{height} = \Delta x \cdot f(x_i).$$

Our approximate area A_n is the sum of the areas of all of these rectangles:

$$\begin{aligned} A_n &= f(x_1)\Delta x + f(x_2)\Delta x + \cdots + f(x_n)\Delta x \\ &= \sum_{i=1}^{n} f(x_i)\Delta x. \end{aligned}$$

Now, we can figure out that

$$\begin{aligned}
\Delta x &= \frac{1}{n} \\
x_i &= (i-1)\Delta x = \frac{i-1}{n} \\
f(x_i) &= (x_i)^2 = \left(\frac{i-1}{n}\right)^2.
\end{aligned}$$

Making these substitutions, we find that

$$\begin{aligned}
A_n &= \sum_{i=1}^{n} \left(\frac{i-1}{n}\right)^2 \cdot \frac{i-1}{n} \\
&= \frac{1}{n^3} \sum_{i=1}^{n} (i-1)^2 \\
&= \frac{1}{n^3} \sum_{i=1}^{n} (i^2 - 2i + 1) \\
&= \frac{1}{n^3} \left(\sum_{i=1}^{n} i^2 - 2\sum_{i=1}^{n} i + \sum_{i=1}^{n} 1 \right).
\end{aligned}$$

As it happens, we have formulas from last time for $\sum i^2$, $\sum i$, and $\sum 1$. Applying these, we find

$$\begin{aligned}
&= \frac{1}{n^3} \left(\frac{n(n+1)(2n+1)}{6} - 2\frac{n(n+1)}{2} + n \right) \\
&= \frac{1}{n^3} \left(\frac{n^3}{3} - \frac{n^2}{2} + \frac{n}{6} \right) \\
&= \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2}.
\end{aligned}$$

Now, it makes sense that lots of narrow rectangles should give a better approximation than a few wide rectangles. Correspondingly, A_n should be a better and better approximation to the area as we make n larger and larger. And in the limit, the area will be

$$\begin{aligned}
\lim_{n\to\infty} A_n &= \lim_{n\to\infty} \frac{1}{3} - \frac{1}{2n} + \frac{1}{6n^2} \\
&= \frac{1}{3}.
\end{aligned}$$

There is an important alternative way to do this: we chose to make the height of the rectangle determined by $f(x_i)$, the value of f on the *left* side of the rectangle. For this particular function, that means our region of "lots of rectangles" always lies *inside* the area we are trying to calculate, so the area A_n

always ought to *underestimate* the area we care about. If A denotes the area we care about, it's intuitive that $A_n \leq A$ for all n, and so

$$\lim_{n\to\infty} A_n \leq A.$$

It is less obvious that these two should in fact be *equal*. Thus, we should probably call $\lim_{n\to\infty} A_n$ by the name $\underline{A}$, the "lower area," as outlined in the first section.

For a more conclusive argument, we could instead take the "upper sum" given by taking the rectangles to intersect the function on their top-*right* corners:

In this case, the area A we care about will be contained in the areas of all the rectangles put together, so we'll have $A \leq \overline{A}_n$ for these $\overline{A}_n$, and consequently

$$A \leq \lim_{n\to\infty} \overline{A}_n.$$

Let's call $\lim_{n\to\infty} \overline{A}_n$ the "upper area" $\overline{A}$. Then A will be sandwiched between $\underline{A}$ and $\overline{A}$. As it turns out, in our case, the two limits $\underline{A}$ and $\overline{A}$ are both equal to $\frac{1}{3}$, so the function $f(x) = x^2$ is "integrable over $[0, 1]$," with "integral" equal to $\frac{1}{3}$.

3 The definite integral

The same procedure can be carried out more generally to give a definition for "integral on an interval $[a, b]$." Some important properties:

- integral as "signed area"
- notation
- linearity
- If a function is continuous on a closed interval $[a, b]$, then it is integrable on $[a, b]$.
- Calculating: Once we know the function is integrable, we only have to calculate a single limit to know the integral. It can be the "upper area," the "lower area," or anything in between.

Assignment 15 (due Wednesday, 15 February)

Section 3.4, Problem 59. This will be graded carefully.

Section 3.6, Problem 28. This will be graded carefully.

Section 3.8, Problems 21, 22, 29, and 35. Problems 22 and 35 will be graded carefully.

Section 4.1, Problems 38, 53, and 54. Problems 38 and 54 will be graded carefully.

Assignment 16 (due Friday, 17 February)

Section 3.4, Problem 14. This will be graded carefully.

Section 3.8, Problems 30 and 36. Problem 36 will be graded carefully.

Section 4.1, Problems 55 and 56. Problem 56 will be graded carefully.

Section 4.2, Concepts Review. Include a picture for number 4.

Section 4.2, Problems 11 and 12. Problem 12 will be graded carefully. (Note: These are very similar to the problems from Section 4.1.)

Math 132, Lecture 16: The Fundamental Theorem of Calculus

Charles Staats

Wednesday, 15 February 2012

1 Accumulation Functions

We are going to show, in this lecture, that in an appropriate sense, "taking definite integrals" is the same thing as "taking antiderivatives." This will explain why we use the "indefinite integral" notation $\int f(x)\,dx$ for the antiderivative of $f(x)$.

There's a problem here, though: a definite integral is a *number*, whereas an antiderivative is a *function*. As I have emphasized repeatedly, *numbers* and *functions* are completely different kinds of things: not so much apples and oranges, as apples and juicers. (A juicer takes one food—say, an apple—and turns it into another food, apple juice. A function takes one number and turns it into another number.) Thus, claiming that the definite integral (a number) is "the same as" the antiderivative (a function) will take some explanation.

In truth, we're not dealing with the definite integral directly. Instead, we use the process of definite integration to produce a new function—the *accumulation function*—that will, it turns out, be equal to an antiderivative.

Definition. Let f be a function that is integrable on $[a, b]$. The *accumulation function* of f is the function F over $[a, b]$ defined by

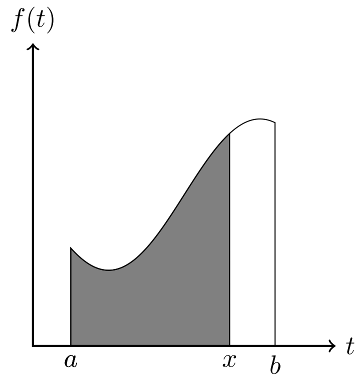
$$F(x) = \int_a^x f(t)\,dt.$$

Note that t here is a "dummy variable;" we could have used any letter other than a or x and it would have worked just as well. We cannot, however, write something like

$$\xcancel{\int_a^x f(x)\,dx},$$

any more than we can say "let x be the function defined by $x = x^2 + 1$." In some sense, this is like a man being his own father. The upper and lower limits (say, a and x) cannot depend on the dummy variable (the thing after the d; in our case, the dummy variable is t).

What does this expression for "accumulation function of f" really mean? Basically, the accumulation function measures the "area under f so far." In the picture below, $F(x) = \int_a^x f(t)\, dt$ is the area of the shaded region.



2 The Fundamental Theorem

Theorem. (The Fundamental Theorem of Calculus)

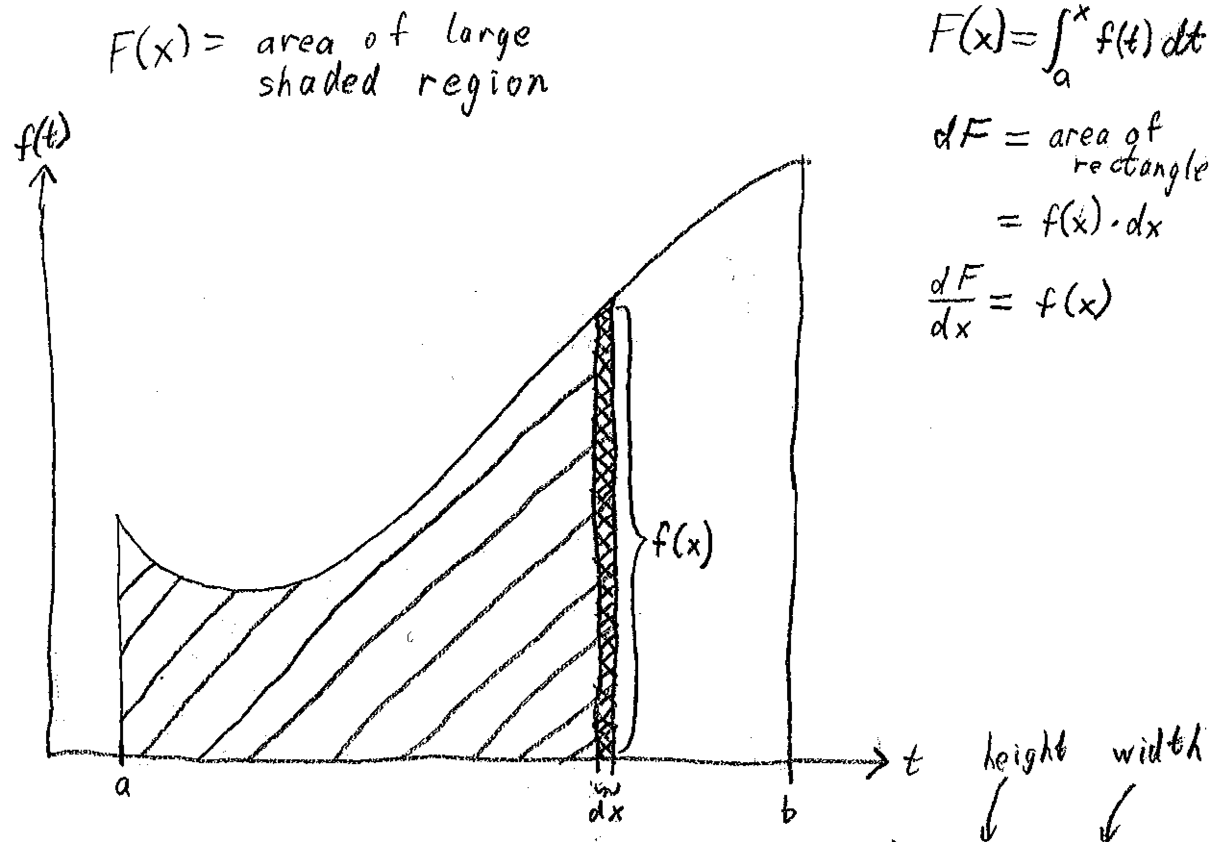
Version 1 Let f be a function continuous on $[a, b]$. Then the accumulation function $\int_a^x f(t)\, dt$ is differentiable, and

$$\frac{d}{dx} \int_a^x f(t)\, dt = f(x).$$

Version 2 Let f be a function continuous on $[a, b]$. Then the accumulation function $\int_a^x f(t)\, dt$ is an antiderivative for f on $[a, b]$.

Sketch of proof. The second version is often more useful, but the first is easier to prove. (Fortunately, they are just slightly different ways of saying exactly the same thing.)

I'm going to sketch how the proof would go in terms of "infinitesimally small" differentials dx and dy.



$$dF = F(x+dx) - F(x) = \text{area of ``rectangle''} = f(x) \cdot dx$$

□

Theoretical significance: If f is continuous on $[a, b]$, then we know it is "integrable," i.e., has a definite integral. Therefore, it has an antiderivative. This is extremely important: for some functions, such as $f(t) = 2^{-t^2}$, the antiderivative is an extremely important function for which there is no formula. (Not just no known formula: no formula period.) But since f is continuous, we know that the antiderivative does, in fact, exist, since it is equal to the integral.

Computational significance: In some situations, we can "reverse" what we know about finding derivatives to find antiderivatives. When this happens (which it often does not), we can use these antiderivatives to compute definite integrals.

Assignment 16 (due Friday, 17 February)

Section 3.4, Problem 14. This will be graded carefully.

Section 3.8, Problems 30 and 36. Problem 36 will be graded carefully.

Section 4.1, Problems 55 and 56. Problem 56 will be graded carefully.

Section 4.2, Concepts Review. Include a picture for number 4.

Section 4.2, Problems 11 and 12. Problem 12 will be graded carefully. (Note: These are very similar to the problems from Section 4.1.)

Assignment 17 (due Monday, 20 February)

Section 4.2, Problems 7, 8, 13, and 14. Problems 8 and 14 will be graded carefully.

Section 4.3, Problems 3, 4, 9, and 10. Problems 4 and 10 will be graded carefully.

Section 4.4, Problems 1 and 2.

Math 132, Lecture 17: Using Antiderivatives to find definite integrals

Charles Staats

Friday, 17 February 2012

1 The method

Recall the Fundamental Theorem of Calculus from last time:

Theorem. (Fundamental Theorem of Calculus) Let f be a function continuous on $[a, b]$. Then the accumulation function $\int_a^x f(t)\ dt$ is an antiderivative for f on $[a, b]$.

The idea for the "practical" application here is the following: when we have somehow managed to compute an antiderivative for f (e.g., using the techniques for finding indefinite integrals), then we can use this to find the accumulation function $A(x) = \int_a^x f(t)\ dt$, which in particular gives us the definite integral since

$$\int_a^b f(t)\ dt = A(b).$$

The "nice" way to do this would be to say, simply, that if we have an antiderivative F, A is equal to F. Unfortunately, this is **not true**, since f will have more than one antiderivative. [We will, however, end up finding out that A is given by $A(x) = F(x) - F(a)$, so in some sense, it is "almost true."]

Exercise 1. If F is an antiderivative of f, then so is the function G defined by $G(x) = x + C$, for any fixed constant C.

However, we were also able to show, using the Mean Value Theorem, that this is the only thing that can go wrong:

Theorem. If F and G are two antiderivatives of f on $[a, b]$, then there is a constant C such that $F(x) = G(x) + C$, for all x.

Now, suppose f is continuous on $[a, b]$, and suppose we have somehow, "magically," produced an antiderivative F for f. Let A be the accumulation function of f, i.e.,

$$A(x) = \int_a^x f(t)\ dt.$$

By the Fundamental Theorem of Calculus, A is also an antiderivative of f. Thus, there is some function constant C such that

$$A(x) = F(x) + C.$$

So, to find A, all we have to do is find C. But this is easy:

$$\begin{aligned} C &= A(x) - F(x) && \text{for all } x\text{. In particular,} \\ C &= A(a) - F(a) \\ &= \left(\int_a^a f(t)\, dt \right) - F(a) \\ &= -F(a), \end{aligned}$$

since $\int_a^a f(t)\, dt = 0$ no matter what f is. ("The area of a vertical line segment is zero.") Thus,

$$\begin{aligned} A(x) &= F(x) - F(a) \\ \int_a^x f(t)\, dt &= F(x) - F(a). && \text{In particular, for } x = b, \\ \int_a^b f(t)\, dt &= F(b) - F(a). \end{aligned}$$

This is what the textbook calls the "Second Fundamental Theorem of Calculus":

Theorem. If f is continuous on $[a, b]$, and F is an antiderivative for f on $[a, b]$, then

$$\int_a^b f(t)\, dt = F(b) - F(a).$$

Personally, I think that when you really understand the first two theorems I stated, and you *really* know what the word "antiderivative" means[1], then you should be able to get to the "Second Fundamental Theorem" without having to remember it as a separate theorem. However, if the name "Second Fundamental Theorem of Calculus" is helpful to you, then by all means use it.

Here's another bit of intuition: recall that the "indefinite integral"

$$\int f(x)\, dx = F(x) + C$$

is just a way of denoting the "antiderivative." You can also think of the "indefinite integral" as an accumulation function for which *we don't know where it starts.* When someone gives us a starting point a and asks for

$$\int_a^x f(t)\, dt,$$

[1] An analogy I used in class last time that may help with this: "F is an antiderivative of f" means precisely the same thing as "f is the derivative of F," much as "John is a child of Jessica" means precisely the same thing as "Jessica is a parent of John."

we now know what C is: it is precisely $-F(a)$ [to ensure that the integral is zero when we set $x = a$]. Thus, we get that

$$\int_a^x f(t)\ dt = F(x) - F(a),$$

where F is given by the indefinite integral. If you will, the indefinite integral has an *indefinite* starting point a; this becomes a definite integral by choosing a definite starting point. With this in mind, I am sometimes inclined to denote the accumulation function by an integral sign with *only* the lower limit: $\int_a^x f(t)\ dt = \int_a f(x)\ dx = F(x) - F(a)$. I probably won't take off points if you use this notation, but please don't tell anyone else I let you do this.

2 Some examples

At this point, we will do some examples, to show how to use the Fundamental Theorem to compute definite integrals in practice (using antiderivatives). Each of the following examples should be done in two steps:

1. First, find the indefinite integral,

$$\int f(x)\ dx = F(x) + C.$$

2. Then, find the definite integral as

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

Example 2. $\displaystyle\int_0^1 t^2\ dt$

Solution.

$$\int x^2\ dx = \tfrac{1}{3}x^3 + C$$

$$\begin{aligned}\int_0^1 t^2\ dt &= \tfrac{1}{3}(1)^3 - \tfrac{1}{3}(0)^3 \\ &= \tfrac{1}{3} - 0 \\ &= \tfrac{1}{3}.\end{aligned}$$

□

Notice how much easier that was than the "limit of $\sum$'s" version.

Example 3. $\displaystyle\int_0^{\pi/3} \sin 3t\; dt$ (Hint: use u-substitution.)

Solution. First, to find $\int \sin 3t\; dt$, let

$$\begin{aligned} u &= 3t \\ du &= 3dt \\ \tfrac{1}{3}du &= dt. \end{aligned}$$

Thus,

$$\begin{aligned} \int \sin 3t\; dt &= \int \sin u \cdot \tfrac{1}{3} du \\ &= \frac{1}{3} \int \sin u\; du \\ &= \tfrac{1}{3}(-\cos u + C_0) \\ &= -\tfrac{1}{3}\cos u + C \\ &= -\tfrac{1}{3}\cos 3t + C, \end{aligned}$$

and so

$$\begin{aligned} \int_0^{\pi/3} \sin 3t\; dt &= -\tfrac{1}{3}\cos 3t\Big|_{t=0}^{\pi/3} \\ &= -\tfrac{1}{3}\cos 3(\pi/3) + \tfrac{1}{3}\cos 3(0) \\ &= -\tfrac{1}{3}\cos\pi + \tfrac{1}{3}\cos 0 \\ &= -\tfrac{1}{3}(-1) + \tfrac{1}{3}(1) \\ &= \tfrac{1}{3} + \tfrac{1}{3} \\ &= \tfrac{2}{3}. \end{aligned}$$

□

Example 4. (Example 12, p. 284 in the textbook) $\int_0^1 \frac{(x+1)\,dx}{(x^2+2x+6)^2}$. (This may require a somewhat clever choice of u.)

Assignment 17 (due Monday, 20 February)

Section 4.2, Problems 7, 8, 13, and 14. Problems 8 and 14 will be graded carefully.

Section 4.3, Problems 3, 4, 9, and 10. Problems 4 and 10 will be graded carefully.

Section 4.4, Problems 1 and 2.

Assignment 18 (due Wednesday, 22 February 22)

Section 4.2, Problems 15 and 16. Calculate each of these integrals two ways: first, using the definition of definite integral (i.e., as a limit of $\sum$'s); second, using the Fundamental Theorem of Calculus. All of these will be graded carefully.

Section 4.4, Problems 3, 4, 15, 16, 35, and 36. The even-numbered problems will be graded carefully.

Section 4.4, Problem 71. This will be discussed in tutorial on Tuesday, but you will get more out of the discussion if you try to solve it yourself ahead of time.

Math 132, Lecture 18: More on finding definite integrals; Average values

Charles Staats

Monday, 20 February 2012

1 Minor shortcuts in taking definite integrals

Recall the basic strategy for finding a definite integral $\int_a^b f(t)\ dt$ using the Fundamental Theorem of Calculus:

1. Find the indefinite integral $\int f(x)\ dx = F(x) + C$.
2. Evaluate the definite integral as

$$\int_a^b f(t)\ dt = F(b) - F(a).$$

While this process is simple to describe, it can take time and space to write out in full. In this section, we will be discussing some "notational shortcuts" that can make the process faster. These "shortcuts" are completely unnecessary for being able to do problems on my tests, but they may help you to finish the test on time.

The first is something that I actually introduced last time in a very casual way. It is just a notation that allows us to avoid writing out the "$F(x) + C$" bit, and skip directly to evaluating the definite integral.

Notation. For any function F, the two expressions

$$F(x)\Big|_{x=a}^{b} \qquad \text{and} \qquad [F(x)]_{x=a}^{b}$$

will both be taken to mean

$$F(b) - F(a).$$

With this notation, we can rewrite the Fundamental Theorem as follows:

$$\int_a^b f(t)\ dt = \left[\int f(t)\ dt\right]_{t=a}^{b}$$

In practice, when evaluating a definite integral, we sometimes use this notation to avoid ever writing the indefinite integral as such.

Example 1. Evaluate $\int_{-1}^{1} t^2 \, dt$.

Solution.

$$\begin{aligned}\int_{-1}^{1} t^2 \, dt &= \left[\tfrac{1}{3}t^3\right]_{t=-1}^{1} \\ &= \tfrac{1}{3}(1)^3 - \tfrac{1}{3}(-1)^3 \\ &= \tfrac{1}{3} - \left(-\tfrac{1}{3}\right) \\ &= \tfrac{2}{3}.\end{aligned}$$

□

Notice how short the solution is with this notation. If I had wanted, I probably could have written it out in one line.

The second "new" trick is a bit more substantial. It allows us to shorten the process of applying u-substitution with definite integrals. When we do this process in full to an indefinite integral like $\int f(g(x))g'(x) \, dx$, we have steps like the following:

(a) Substitute in $g(x) = u$ and $g'(x) \, dx = du$. The integral above then becomes $\int f(u) \, du$.

(b) Evaluate this indefinite integral as $F(u) + C$.

(c) Substitute back in $u = g(x)$, obtaining $F(g(x)) + C$.

Consequently, if $\int f(u) \, du = F(u) + C$, then $\int f(g(x))g'(x) \, dx = F(g(x)) + C$. Thus, if we want to find the definite integral $\int_a^b f(g(x))g'(x) \, dx$, we obtain

$$\begin{aligned}\int_a^b f(g(x))g'(x) \, dx &= F(g(x))\big|_{x=a}^{b} \\ &= F(g(b)) - F(g(a)) \\ &= F(u)\big|_{u=g(a)}^{g(b)} \\ &= \int_{g(a)}^{g(b)} f(u) \, du.\end{aligned}$$

This can be simpler and quicker to write, because applying g to the particular numbers a and b can give much shorter answers than taking a formula for $g(x)$ and plugging it into F, simplifying, and *then* plugging in the numbers a and b.

Example 2. (Example 12, p. 248 in the textbook; repeated from last lecture) Evaluate

$$\int_0^1 \frac{(x+1) \, dx}{(x^2 + 2x + 6)^2}.$$

Solution. Set

$$u = x^2 + 2x + 6 = g(x)$$
$$du = (2x + 2)dx = 2(x + 1)\ dx.$$

Note that when $x = 0$, $u = g(0) = 6$; and when $x = 1$,

$$u = g(1) = 1 + 2 + 6 = 9.$$

Thus,

$$\begin{aligned}
\int_0^1 \frac{(x+1)\ dx}{(x^2+2x+6)^2} &= \int_6^9 \frac{\frac{1}{2}du}{u^2} \\
&= \frac{1}{2}\int_6^9 u^{-2}\ du \\
&= \frac{1}{2}\left[\frac{u^{-1}}{-1}\right]_{u=6}^{9} \\
&= \frac{1}{2}\left[\frac{-1}{u}\right]_{u=6}^{9} \\
&= \frac{1}{2}\left(\frac{-1}{9} - \frac{-1}{6}\right) \\
&= \frac{1}{2}\cdot\frac{1}{3}\left(\frac{-1}{3} - \frac{-1}{2}\right) \\
&= \frac{1}{6}\cdot\frac{-2-(-3)}{6} \\
&= \frac{1}{6}\cdot\frac{1}{6} \\
&= \frac{1}{36}.
\end{aligned}$$

□

In the version of this solution that I typed up, a lot of the space is devoted to adding fractions. The actual calculus is only a few lines at the beginning.

2 Using symmetry

This section discusses, quickly, some tricks that can simplify integration in some very special circumstances. Surprisingly enough, these very special circumstances do in fact show up in practice, so the "symmetry tricks" are worth knowing.

Warning. *To apply the tricks below to $\int_a^b f(t)\ dt$, you have to look at a and b,*

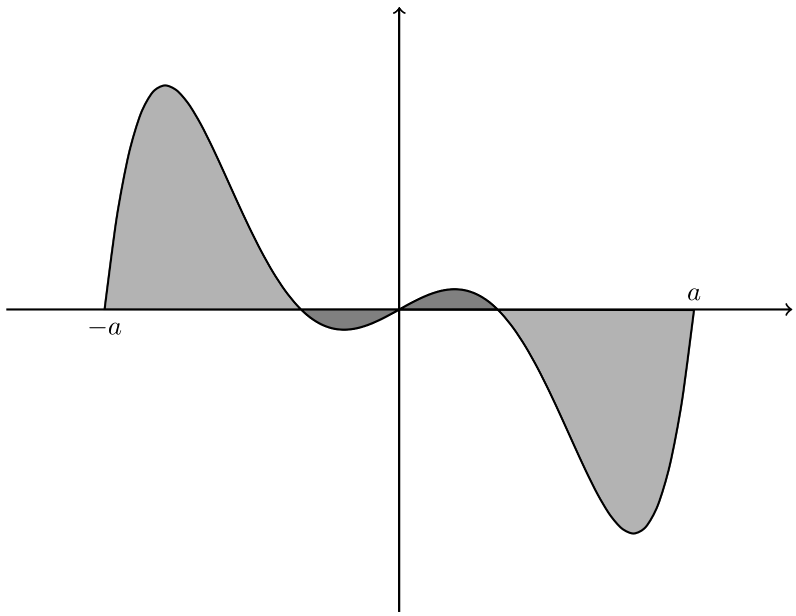


Figure 1: For an odd function, the area on the left negates the area on the right, so the sum is zero.

not just f.

Theorem. (Odd functions) If f is an odd function (i.e., $f(-x) = -f(x)$ for all x), then

$$\int_{-a}^{a} f(t)\ dt = 0.$$

Note that this only applies if we are integrating from $-a$ to a. The idea is that the "area on the left" cancels the "area on the right."

Theorem. (Even functions) If f is an even function (i.e., $f(-x) = f(x)$ for all x), then

$$\int_{-a}^{a} f(t)\ dt = 2\int_{0}^{a} f(t)\ dt.$$

In the case of an even function, the "area on the left" equals the "area on the right," so

$$\int_{-a}^{a} f(t)\ dt = \underbrace{\int_{-a}^{0} f(t)\ dt}_{\text{area on left}} + \underbrace{\int_{0}^{a} f(t)\ dt}_{\text{area on right}} = 2\int_{0}^{a} f(t)\ dt.$$

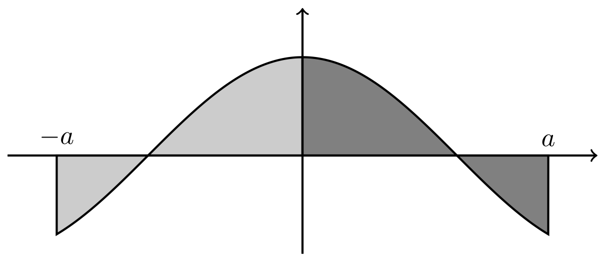


Figure 2: For an even function, the area on the left equals the area on the right.

Example 3. Evaluate $\int_{-4}^{4} (x^3 + x^2)\, dx$.

Solution. Note that x^3 is odd and x^2 is even. Thus, we have

$$\begin{aligned}\int_{-4}^{4} (x^3 + x^2)\, dx &= \int_{-4}^{4} x^3\, dx + \int_{-4}^{4} x^2\, dx \\ &= 0 + 2\int_{0}^{4} x^2\, dx \\ &= 2\left[\tfrac{1}{3}x^3\right]_{x=0}^{4} \\ &= 2 \cdot \tfrac{1}{3}(4)^3 \\ &= \frac{128}{3}.\end{aligned}$$

□

3 Average value of a function

It is a common theme in calculus that much of what can be done with sums can also be done with integrals. The definition of "average value" is a case in point.

When we take the average of a bunch of numbers $x_1, \ldots, x_n$, we are trying, in some sense, to approximate the entire data set by a single number. More precisely, we are finding the single number M such that if we pretended all of the x_i were equal to M, they would still have the correct sum:

$$\begin{aligned}\sum_{i=1}^{n} M &= \sum_{i=1}^{n} x_i \\ nM &= \sum_{i=1}^{n} x_i \\ M &= \frac{1}{n}\sum_{i=1}^{n} x_i.\end{aligned}$$

This gives us a formula for the average M: you take the sum of all the x_i, and then divide this sum by the number of data points.

When we take the average of a function f over an interval $[a, b]$, we are trying to approximate the entire function by a single number. More precisely, we are finding a single number M such that if we pretended $f(x)$ were equal to M for every x in the interval, we would still obtain the correct integral:

$$\begin{aligned} \int_a^b M\ dx &= \int_a^b f(x)\ dx \\ M(b-a) &= \int_a^b f(x)\ dx \\ M &= \frac{1}{b-a} \int_a^b f(x)\ dx. \end{aligned}$$

Geometrically, the average value M is the height of a rectangle over $[a, b]$ that has the same area as the area under f.

There is a Mean Value Theorem for integrals stating that if f is continuous on $[a, b]$, then f attains its average value at some point in (a, b). In other words, there exists c in (a, b) such that

$$f(c) = \frac{1}{b-a} \int_a^b f(x)\ dx.$$

I may eat these words later, but for the moment, I don't see any particular reason for you to need to remember this. It would, however, be good to understand how this is very closely related to the last Mean Value Theorem for derivatives, using the Fundamental Theorem of Calculus.

Assignment 18 (due Wednesday, 22 February 22)

Section 4.2, Problems 15 and 16. Calculate each of these integrals two ways: first, using the definition of definite integral (i.e., as a limit of $\sum$'s); second, using the Fundamental Theorem of Calculus. All of these will be graded carefully.

Section 4.4, Problems 3, 4, 15, 16, 35, and 36. The even-numbered problems will be graded carefully.

Section 4.4, Problem 71. This will be discussed in tutorial on Tuesday, but you will get more out of the discussion if you try to solve it yourself ahead of time.

Assignment 19 (due Friday, 24 February)

Section 4.4, Problems 5, 6, 17, 18, 37, and 38. The even-numbered problems will be graded carefully.

Section 4.5, Problems 1, 2, 35, and 36. The even-numbered problems will be graded carefully.

Explain how the Fundamental Theorem of Calculus relates the Mean Value Theorem for Derivatives to the Mean Value Theorem for Integrals. This will be discussed in tutorial on Thursday, but you will get more out of the discussion if you try to solve it yourself ahead of time.

Test Wednesday, 29 February

The text will be cumulative, but will be focused on (Lecture 7, section 2) through Lecture 18. See also Assignments 8–19. The most relevant sections in the textbook are 3.3–3.6, 3.8, and 4.1–4.5, with particular emphasis on 3.4 (practical problems) and 4.4 (evaluating integrals).

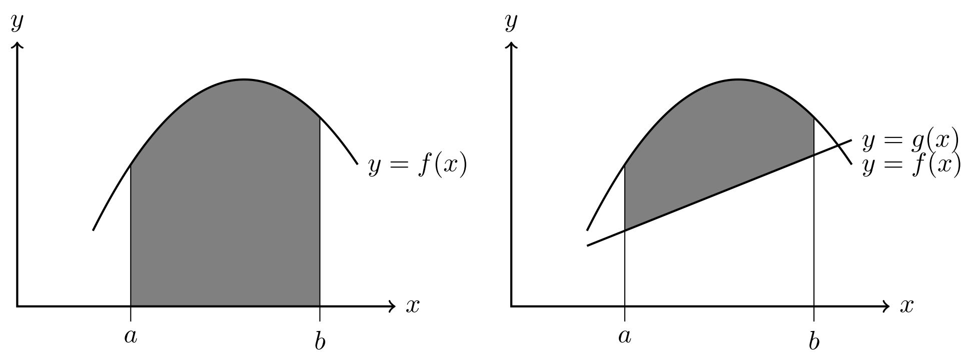
Math 132, Lecture 19: Area between two curves

Charles Staats

Wednesday, 22 February 2012

1 Area between two curves

We introduced the definite integral as a way to define and calculate the area under the graph of a function. In this lecture, we will discuss how to use the same mathematical tools to solve a problem that, on the surface, appears harder: finding the area between two such graphs. The "slogan" that the book gives for



(a) area under $y = f(x)$ from $x = a$ to $x = b$ $\int_a^b f(x)\, dx$

(b) area between $y = f(x)$ and $y = g(x)$ from $x = a$ to $x = b$

finding such areas is "slice, approximate, and integrate." In other words,

1. Find an expression for dA (an infinitesimal change in the area, typically by a very thin rectangle—a "slice") in terms of dx (an infinitesimal change in x; typically, the width of the rectangle).

2. Once we have $dA = \text{stuff}(x)\, dx$, we "add up the infinitely many infinitesimally small rectangles"—in other words, integrate:

$$A = \int_a^b \text{stuff}(x)\, dx.$$

Often, this will end up taking the form

$$A = \int_a^b (\text{top function} - \text{bottom function})\, dx.$$

> Sometimes it's obvious what a and b are, but sometimes you need to compute them by taking the intersection points of the top function and the bottom function; in other words, solving $f(x) = g(x)$ for x.

One unstated, but often crucial, first step is to sketch the region first and make sure you understand exactly what is going on. Let's do some examples to illustrate the method.

Example 1. Find the area between $y = 3 - x^2$ and $y = x$ from $x = -1$ to $x = 1$.

Solution.

□

Example 2. Find the area between $y = x^2$ and $y = x^3$ to the right of $x = 0$.

Solution.

□

Example 3. Find the area between $y = \sin x$ and $y = 0$ from $x = -\pi$ to $x = 0$.

Solution.

□

2 What is a solid of revolution?

Assignment 19 (due Friday, 24 February)

Section 4.4, Problems 5, 6, 17, 18, 37, and 38. The even-numbered problems will be graded carefully.

Section 4.5, Problems 1, 2, 35, and 36. The even-numbered problems will be graded carefully.

Explain how the Fundamental Theorem of Calculus relates the Mean Value Theorem for Derivatives to the Mean Value Theorem for Integrals. This will be discussed in tutorial on Thursday, but you will get more out of the discussion if you try to solve it yourself ahead of time.

When you are preparing for the test, you may want to write down any specific questions you have so that you will remember to ask them on Monday (which I will be devoting mostly to review). As always, you may also come to my office hours or make an appointment to see me. And don't forget that tutorial is taylor-made for asking questions.

Assignment 20 (due Monday, 27 February)

Section 4.4, Problems 19, 20, 39, and 40. The even-numbered problems will be graded carefully.

Section 4.5, Problems 7 and 8. Problem 8 will be graded carefully.

Section 5.1, Problems 3, 4, 19, and 20. All four of these will be graded carefully.

When you are preparing for the test, you may want to write down any specific questions you have so that you will remember to ask them on Monday (which I will be devoting mostly to review). As always, you may also come to my office hours or make an appointment to see me. And don't forget that tutorial is taylor-made for asking questions.

Test Wednesday, 29 February

The test will be cumulative, but will be focused on (Lecture 7, section 2) through Lecture 18. See also Assignments 8–19 and the first part of 20 (anything from Chapter 4). The most relevant sections in the textbook are 3.3–3.6, 3.8, and 4.1–4.5, with particular emphasis on 3.4 (practical problems) and 4.4 (evaluating integrals).

Math 132, Lecture 21: Prelude to logarithms and exponentials

Charles Staats

Monday, 27 February 2012

1 Test Wednesday

In case you had forgotten, there is a test coming up on Wednesday (next class meeting). I expect to devote most of this class period to review, but I thought I would put in a bit of lecture to prepare you for the fairly non-intuitive approach to logarithms and exponentials we'll be starting on Friday.

2 The dog and the squirrel

There is a well-known experiment that goes something like this. A dog is placed on a leash near a bowl of food. However, the leash is anchored so that the dog must take a detour away from the food and around a post in order to get to the food. If it tries to go toward the food directly, the leash will pull it up short. Experiments show that dogs are very bad at solving this problem; they will strain toward the bowl, but not think to go away from it to unloop the leash.

By contrast, if a squirrel is placed in the same situation, it will very quickly take the detour and solve the problem—in spite of the fact that by most measures, squirrels are less intelligent than dogs. This is because a squirrel is accustomed to traveling through the treetops, where indirect routes are the norm. A squirrel wanting to go from point A to point B cannot simply jump directly toward point B; it must figure out a path of smaller jumps and runs along branches.

A lot of mathematics is like this. We may understand exactly where it is we want to go, or what it is we want to prove; but if we try to prove it directly, we can't get there. Instead, we have to think like the squirrel and take an indirect route. We start off going in a strange direction that may seem to have nothing to do with our goal, but will end up "unlooping" our "leashes" from any number of obstacles that would otherwise bring us up short.

3 Exponentials and logarithms

One such problem is producing a "good" definition for 2^x. If x is a rational number—say, $x = p/q$ where p and q are integers—then we can define

$$2^{p/q} = \sqrt[q]{2^p}.$$

This is the only definition that fulfills the standard rules of exponents:

$$\left(2^{p/q}\right)^q = 2^{\frac{p}{q} \cdot q} = 2^p,$$

i.e., $2^{p/q}$ is a q^{th} root of 2. Unfortunately, we cannot use this to define 2^π, since π cannot be written as a fraction. We could try something like

$$2^\pi = \lim_{r \to \pi} 2^r,$$

where we take the limit over *rational numbers only.* In principle, this method might work. In practice, it will prove very difficult to show directly that the limit exists and that 2^x, when defined this way, will have all the properties we expect (e.g., 2^x is what we have already said it should be when x is rational, 2^x is continuous, $2^{x+y} = 2^x \cdot 2^y$, etc.). Instead, we will take a "squirrel approach", starting out in a direction that seems wrong but will ultimately lead to a better solution.

When I start out next lecture by defining the "natural logarithm" as the "antiderivative of $1/x$," you may be confused because this seems to have nothing to do with the logarithm you have seen before. Remember that we are taking the "squirrel approach." This logarithm will turn out to be the same as the one you know, but if we started out with the definition you know, we'd be heading "directly toward the target"—and we'd never get there.

Test Wednesday, 29 February

The test will be cumulative, but will be focused on (Lecture 7, section 2) through Lecture 18. See also Assignments 8–19 and the first part of 20 (anything from Chapter 4). The most relevant sections in the textbook are 3.3–3.6, 3.8, and 4.1–4.5, with particular emphasis on 3.4 (practical problems) and 4.4 (evaluating integrals).

Assignment 21 (due Friday, 2 March)

Section 5.1, Problems 36 and 37. These will be discussed in tutorial on Thursday.

Section 5.3, Concepts Review, all four questions (but not the second blank in the second question). These will be graded carefully. In particular, if you give only the answers without any work, you will not receive credit.

Math 132, Lecture 23: Logarithms, inverse functions, and exponentials

Charles Staats

Monday, 5 March 2012

1 Algebraic properties of the natural logarithm

Recall that, taking the "squirrel approach" (i.e., indirect approach that is faster in the long run), we defined the natural logarithm function ln by

$$\ln x = \int_1^x \frac{dt}{t}.$$

Note that $\ln x$ is only defined when x is positive, since we cannot integrate $1/t$ across a point where it is not only undefined, but also unbounded. The graph of the natural logarithm is shown in Figure 1.

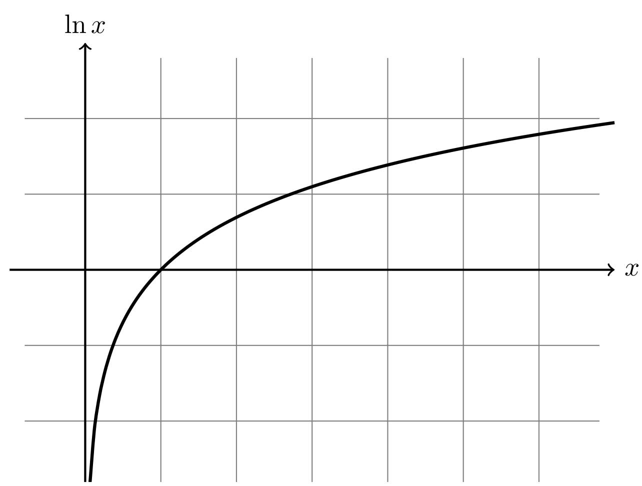


Figure 1: Graph of the natural logarithm.

At the end of the last lecture, I had just stated the following important algebraic properties of the natural logarithm function:

(i) $\ln 1 = 0$.

(ii) $\ln ab = \ln a + \ln b$ for any $a, b > 0$.

(iii) $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$ for any $a, b > 0$.

(iv) $\ln(a^r) = r \ln a$ for any $a > 0$ and any rational number r (so that $a^r = a^{p/q} = \sqrt[q]{a^p}$ is defined).

We are about to prove these.

Note that these are all statements about a function (the natural logarithm) that, right now, we "know" (i.e., have proven) very little about. Practically the only thing we know how to do is differentiate it. In situations like this, the following strategy can often be helpful for showing identities of the form $f(x) = g(x)$:

1. Show that $f'(x) = g'(x)$, and that these are "nice" functions. Since f and g are antiderivatives of the same function, they must differ by a constant: $f(x) = g(x) + C$.

2. Show that $f(x_0) = g(x_0)$ at some point x_0. Thus, in the case $x = x_0$, we have $f(x_0) = g(x_0) + C$ and $f(x_0) = g(x_0)$, so the constant C is equal to zero. Thus, $f(x) = g(x)$ for all x.

I may ask you to repeat the strategy above on the final exam. Now, we will apply the strategy in the proofs of (ii) and (iv) below.

Proof. (i) Showing that $\ln(1) = 0$ is easy:

$$\ln(1) = \int_1^1 \frac{dt}{t} = 0$$

since the "area" of a vertical line segment is always zero.

(iv) We will use the strategy to show that for fixed rational r, the two functions $\ln(x^r)$ and $r \ln x$ are equal.

First, we show that they have the same derivative:

$$\frac{d}{dx}\ln(x^r) = \frac{1}{x^r}\frac{d}{dx}x^r = \frac{1}{x^r} \cdot r x^{r-1} = \frac{r}{x}$$
$$\frac{d}{dx} r \ln x = r \frac{d}{dx}\ln x = r \cdot \frac{1}{x} = \frac{r}{x}.$$

Since $\ln(x^r)$ and $r \ln x$ are both antiderivatives of r/x on the interval $(0, \infty)$, we know that they differ by a constant C:

$$\ln(x^r) = r \ln x + C.$$

Second, we determine what C is, by evaluating the two functions at the one x-value we know: $x = 1$. We find that

$$\ln(1^r) = \ln 1 = 0$$
$$r \ln 1 = r \cdot 0 = 0.$$

Thus, we have

$$\begin{aligned} \ln(1^r) &= r \ln 1 + C \\ 0 &= 0 + C \\ C &= 0. \end{aligned}$$

Hence, from the first step,

$$\ln(x^r) = r \ln x.$$

(ii) We again use the strategy to show that $\ln ab = \ln a + \ln b$. Unfortunately, in this case, it is even less obvious how to turn the two sides into functions of x. As it turns out, we can make it work by fixing a and setting $x = b$. Thus, we are trying to show that $\ln ax = \ln a + \ln x$.

First, we show that the derivatives of both sides are equal:

$$\begin{aligned} \frac{d}{dx} \ln ax &= \frac{1}{ax} \frac{d}{dx}(ax) = \frac{1}{ax} \cdot a = \frac{1}{x} \\ \frac{d}{dx} \ln a + \ln x &= 0 + \frac{1}{x} = \frac{1}{x}. \end{aligned}$$

Thus, $\ln ax$ and $\ln a + \ln x$ differ by a constant.

Second, to show that this constant is zero, we evaluate both sides at $x = 1$:

$$\begin{aligned} \ln(a \cdot 1) &= \ln a \\ \ln a + \ln 1 &= \ln a + 0 = \ln a. \end{aligned}$$

Thus, the constant is zero, and so

$$\ln ax = \ln a + \ln x.$$

In particular, setting $x = b$, we find that

$$\ln ab = \ln a + \ln b.$$

(iii) Finally, given that we have already shown (ii) and (iv), the proof that $\ln(a/b) = \ln a - \ln b$ can be done purely "algebraically," with no need for calculus:

$$\begin{aligned} \ln\left(\frac{a}{b}\right) &= \ln(a \cdot b^{-1}) \\ &= \ln a + \ln b^{-1} \\ &= \ln a + (-1) \ln b \\ &= \ln a - \ln b. \end{aligned}$$

□

Recalling our original mission to seek a good definition for a^x and the associated function $\log_a x$, let's define the latter by

$$\log_a x := \frac{\ln x}{\ln a}$$

for any $a > 0$. It is not hard to do the algebra to show that properties (i)–(iv) all hold for $\log_a$ and not just ln. In particular, property (iv) will tell us that

$$\begin{aligned}\log_a a^r &= \frac{\ln a^r}{\ln a} \\ &= \frac{r \ln a}{\ln a} \\ &= r\end{aligned}$$

whenever r is rational. Since $\log_a a^x = x$ is perhaps the defining property of the base-a logarithm you may have seen before, this suggests that we are on the right track.

2 Inverse functions

In particular, if you accept that the definition we have given above for $\log_a$ really is the "right" definition, then we can turn the usual equality around and define exponential by using the logarithm, rather than the other way around:

Definition. (Preliminary) If a is a positive number and x is any real number, we define a^x to be the solution y to the equation

$$\log_a y = x.$$

Note that this definition is constructed precisely so that $\log_a a^x = \log_a y = x$. Unfortunately, there are two issues that need to be addressed before we can make this definition for real:

- Does the equation have a solution?
- Does the equation have only one solution?

In other words, we want to make sure that a^x exists and is unambiguous. It is a similar situation to $\sqrt{x}$. You cannot simply define $\sqrt{x}$ to be "the solution to the equation $y = x^2$," since this equation has two solutions if $x > 0$ (one positive and one negative), and no solutions if $x < 0$. Instead, you define $\sqrt{x}$ to be "the *nonnegative* solution to the equation $y = x^2$ if $x \geq 0$ and undefined otherwise," thus eliminating the ambiguity and being upfront about the fact that $\sqrt{x}$ is only defined for nonnegative x.

To see how this is done in general (so that we can apply it to our special case to define a^x), we will discuss *inverse functions.*

Inverse functions are one of those ideas in mathematics that seem very natural in principle (one function gives the solution to another function, much as $y = \sqrt{x}$ gives a solution to $x = y^2$), but the notation can add extra confusion. I realized when I was trying to type this up that I would like to take some more time to figure out a way of presenting it that clarifies rather than obfuscates the ideas. Thus, I have put "inverse functions" on hold until the next lecture. In the mean time, let's look at logarithmic differentiation.

3 Logarithmic differentiation

One of the key properties of the logarithm is that it turns the “hard” operations of multiplication and division into the comparatively “easy” operations of addition and subtraction. When the logarithm was first invented (according to the textbook), it was used to simplify long, complicated by-hand calculation. Now that calculators are available, this particular usage is moot. But the conversion of multiplication to addition can still be useful to simplify advanced computations, like taking derivatives.

Example 1. Let $y = \dfrac{\sqrt{1+x^2}}{(x+1)^{2/3}}$. Differentiate y with respect to x.

Solution. This is certainly possible without any use of logarithms: we can combine the quotient rule, the power rule, and the chain rule for a rather messy solution. But let's look at what we can do by taking logarithms, and *then* differentiating.

$$\ln y = \tfrac{1}{2}\ln(1+x^2) - \tfrac{2}{3}\ln(x+1)$$

□

Assignment 23 (due Wednesday, 7 March)

Even-numbered problems will be graded carefully. Problems in [brackets] need not be handed in, but should be done as practice for the final exam.

[Section 4.4, Problems 21–28.]

Section 5.1, Problems 21 and 22.

Section 5.3, Problems 5–8.

[Section 5.3, Problems 9–12.]

Section 6.1, Problems 7–10 and 19–22.

[Section 6.3, Problems 3–6, 11–14, and 37–40.]

Exam Wednesday, March 14

The exam will cover all the material from this quarter, including the lecture on Wednesday, March 7. I may also include differentiation questions that could have been asked last quarter; however, except for these, I will not design any questions specifically to test material from last quarter.

The exam will take place at 10:30AM in the same room as the lectures.

Math 132, Lecture 24: Exponential functions

Charles Staats

Wednesday, 7 March 2012

1 Logistics

- Seth will be holding a review session tomorrow (Thursday) at the usual tutorial time (noon–1:20) in Eckhart 207. I encourage you all to go, whether or not you are in Seth's tutorial. Bring your own questions, or listen to those of your peers. Questions might include homework questions or test questions that you did not fully understand. **NOTE:** Seth does not know what will be on the final exam; don't ask him.
- I will be holding class at the usual time and place on Friday. I will be reviewing rather than introducing new material. Attendance is optional but encouraged. Please bring your questions.
- Time changes on Sunday. If you forget to change your clock, you may be an hour **LATE** to your first exam. Don't forget.
- The final exam will be Wednesday, March 14, at 10:30AM, in the same room as the lectures.

2 Differentiating exponential functions

Before I get into the theory, I want to make absolutely sure I cover the things in this lecture for which you will be responsible on the final exam.

- For any $a > 0$, the exponential a^x is defined for all x. Again: a must be positive, but x can be any real number—$2, 0, -7/2, \pi - \pi^2, \ldots$.[1]
- $\dfrac{d}{dx}a^x = (\ln a) \cdot a^x$

[1] For certain values of x, you can define a^x when $x \leq 0$. For instance, if x is a positive integer, a^x is defined for all a. But you don't have a continuous function a^x defined for all x unless $a > 0$. For instance,

$0^{-1} = \frac{1}{0}$ is undefined $(a = 0, x = -1)$.

$(-1)^{1/2} = \sqrt{-1}$ is undefined $(a = -1, x = \frac{1}{2})$.

- There is a unique number, called e (the "Euler number"), such that $\ln e = 1$. Consequently, e^x is its own derivative:

$$\frac{d}{dx}e^x = (\ln e) \cdot e^x = 1 \cdot e^x = e^x.$$

  The function e^x is sometimes denoted $\exp(x)$ and called the "natural exponential function." You don't need to memorize the digits of e for the exam, but since I've said this much, I might as well add that e is an irrational number and its first few digits are

$$e \approx 2.718281828459045$$

Example 1. Find the derivative of the function f defined by

$$f(x) = 3^{2x}.$$

Solution.

$$\begin{aligned} f'(x) &= (\ln 3) \cdot 3^{2x} \cdot \frac{d}{dx}(2x) \\ &= (\ln 3) \cdot 3^{2x} \cdot 2 \\ &= (2 \ln 3) \cdot 3^{2x} \end{aligned}$$

□

Exercise 2. (optional) Redo the exercise above, first using the fact that

$$f(x) = 3^{2x} = (3^2)^x = 9^x.$$

Explain why the two answers are the same. (Hint: use the algebraic properties of ln.)

Note that if you can differentiate 9^x (this exercise), but cannot differentiate 3^{2x} using the Chain Rule (previous Example), you are not prepared for the test.

3 Defining exponentials

Mathematicians have this weird hangup about telling people rules (e.g., rules for computing derivatives) without telling them why these rules work. Thus, in what time is left to us, I will try to explain how to finish up the "squirrel route" to get a rigorous definition of exponentials and compute their derivatives.

Recall that we defined the natural logarithm ln by

$$\ln x = \int_1^x \frac{dt}{t}$$

and the base-a logarithm $\log_a$ by

$$\log_a x = \frac{\ln x}{\ln a}$$

whenever $a > 0$ and $a \neq 1$. We then showed the following, whenever r is a rational number:

$$y = a^r \implies \log_a y = r. \tag{1}$$

So far, we have only really defined a^x when x is rational, via

$$a^{p/q} = \sqrt[q]{a^p}.$$

We're going to use the property (1) to define a^x whenever x is real. Thus, we'll be able to talk about $a^\pi, a^{-\sqrt{2}}$, etc. More precisely, we're going to define the function $f(x) = a^x$ by the rule

$$f(x) \text{ is the number } y \text{ such that } \log_a y = x.$$

Or, in other words,

$$y = a^x \iff \log_a y = x.$$

But we need to make sure that this "rule" actually defines a function: we need to show that

(i) For any x, there is a number y (which we will call a^x) such that $\log_a y = x$.

(ii) a^x is unambiguous: i.e., for each x, there is only one number y such that $\log_a y = x$.

To understand these properties better (and prove them), we will bring in the graphs.

4 The graphs

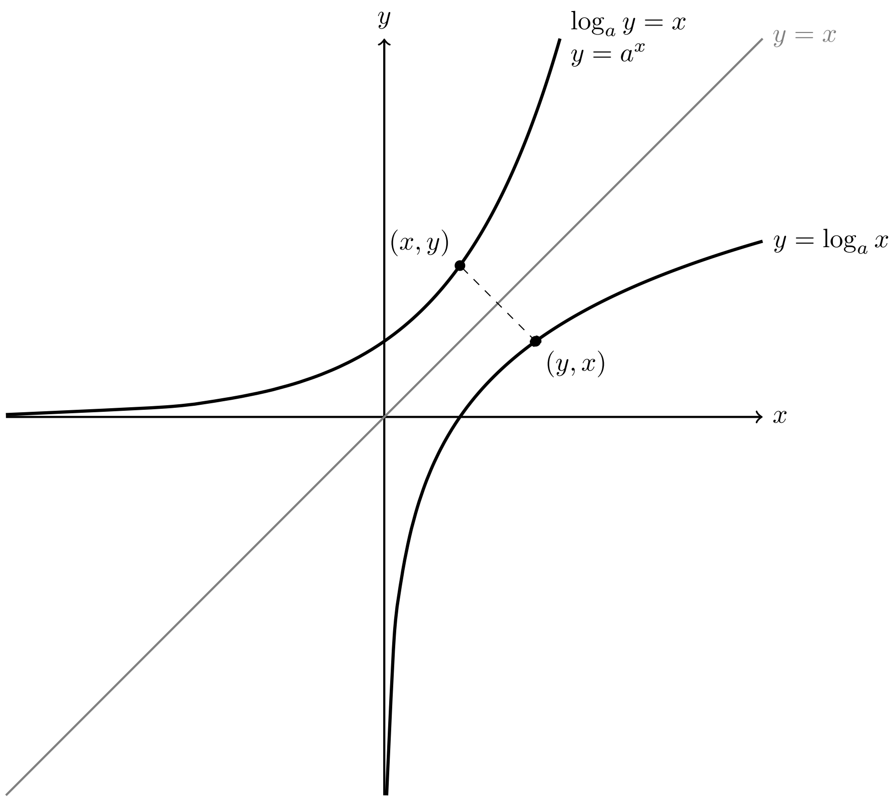
Recall that the graph of the equation $y = \log_a x$ consists, by definition, of

the set of all points (x, y) such that $y = \log_a x$.

Correspondingly, the graph of $y = a^x$ should be

$$\begin{aligned} &\text{the set of all points } (x, y) \text{ such that } y = a^x \\ &\quad = \text{the set of all points } (x, y) \text{ such that } \log_a y = x \\ &\quad = \text{the set of all points } (x, y) \text{ such that } x = \log_a y. \end{aligned}$$

If we compare this to the graph of $y = \log_a x$, we we see that the roles of x and y are switched. In other words, (x, y) lies on the graph of $y = a^x$ if and only if (y, x) lies on the graph of $y = \log_a x$. Geometrically, this corresponds to reflecting about the line $y = x$:



Thus, what we need to do is to show that the graph of

$$\log_a y = x$$

does in fact give y as a function of x. In other words, that every vertical line meets this graph in exactly one place. And (again) switching the role of x and y (= reflecting about the line $y = x$), we realize this is equivalent to showing that every *horizontal* line meets the graph of $y = \log_a x$ in exactly one place.

To show this, the key fact we need is that $y = \log_a x$ is continuous and monotonic on $(0, \infty)$.

Claim. The function f defined by $f(x) = \log_a x$ is monotonic and continuous on $x > 0$, i.e., wherever it is defined.

Proof. By definition,

$$\begin{aligned} f(x) &= \frac{\ln x}{\ln a} \\ &= \tfrac{1}{\ln a} \cdot \ln x \\ f'(x) &= \tfrac{1}{\ln a} \frac{d}{dx} \ln x \\ &= \frac{1}{\ln a} \frac{1}{x}. \end{aligned}$$

Thus, $f'(x)$ exists whenever $x > 0$. Consequently, f is continuous, since differentiable functions are continuous. Moreover, for any $x > 0$, $f'(x)$ has the same sign as $\ln a$. Thus, f is increasing on the entire interval $(0, \infty)$ if $\ln a > 0$, and decreasing on the entire interval if $\ln a < 0$. Either way, f is monotonic. □

For simplicity, let's assume $\ln a > 0$, i.e., $a > 1$. Then we have

(i) Every horizontal line meets the graph, by the Intermediate Value Theorem.

(ii) A horizontal line meets the graph in only one place, since f is increasing: to the right of x_0, f lies above the horizontal line. To the left of x_0, f lies below the horizontal line.

Thus, we can in fact define $y = a^x$ by $\log_a y = x$, since we have already defined $\log_a$.

5 Derivatives

To differentiate $y = a^x$, we use implicit differentiation:

$$\begin{aligned} y &= a^x \\ \log_a y &= x \\ \frac{1}{\ln a} \cdot \ln y &= x \\ \frac{1}{\ln a} \cdot \frac{1}{y}\frac{dy}{dx} &= \frac{d}{dx}(x) \\ \frac{1}{(\ln a)y}\frac{dy}{dx} &= 1 \\ \frac{dy}{dx} &= (\ln a)y \\ \frac{dy}{dx} &= (\ln a)a^x. \end{aligned}$$

This is a special case of the "inverse function theorem," which states, roughly, that

$$\frac{dx}{dy} = \frac{1}{dy/dx}.$$

6 The existence of e

Let's show that there exists a (unique) number e such that $\ln e = 1$. We use the intermediate value theorem. We know that $\ln x$ attains the value 0 at $x = 1$. We know that $\ln x$ attains really big values since $\lim_{x\to\infty} \ln x = \infty$. Thus, it must hit every y-value in between. In particular, there exists x such that $\ln x = 1$.

Note: The paragraph above is essentially the same argument as was used for (i): showing that $y = \log_a x$ meets every horizontal line.

Exam Wednesday, March 14

The exam will cover all the material from this quarter, including the lecture on Wednesday, March 7. I may also include differentiation questions that could have been asked last quarter; however, except for these, I will not design any questions specifically to test material from last quarter.

The exam will take place at 10:30AM in the same room as the lectures.